# BLIND ADAPTIVE KRYLOV SUBSPACE MULTIUSER DETECTION

*Antonio J. Caamaño, Daniel Segovia-Vargas and Javier Ramos*

Departamento de Teoría de la Señal y Comunicaciones,
Universidad Carlos III de Madrid,
C/ Butarque 15, Leganés, 28911,
SPAIN

## ABSTRACT

A new method for low-complexity Multiuser Detection (MUD) based in the Fast Subspace Decomposition (FSD) is proposed. The use of FSD allows the estimation of the number of users along with the multiuser detection on line. This leads to a fast multiuser estimation-detection scheme with ultra-low complexity. Furthermore, the method is proved to be strongly consistent and blind. This is applied here to the MMSE. Results included show MMSE performance at a fraction of the computational cost reported until now. The use for UMTS-TDD receivers is also proposed.

## 1. INTRODUCTION

The implementation of advanced Multiuser Detectors in practical hardware devices has been scarce [1] and even testimonial despite the well known advantages of this Signal Processing technique. The main reason has been the complexity involved in such detectors, even in the simplest of them, i.e., linear multiuser detector. Although some attempts have been made to develop low complexity multiuser detectors [2] this is still a ground waiting for major advances. The authors propose here a new scheme for multiuser detection of low comlexity based in the subspace formulation of the decorrelating and MMSE detectors and, mainly, in the Fast Subspace Decomposition technique introduced by Xu and Kailath [3].

## 2. SIGNAL MODEL

Let us consider a baseband direct sequence CDMA ensemble of $K$ users. The received signal can be modeled as

$$r(t) = S(t) + \sigma n(t) \tag{1}$$

where $n(t)$ is white Gaussian noise with unit power spectral density, and $S(t)$ is the superposition of the data signals of the $K$ users, given by

$$S(t) = \sum_{k=1}^{K} A_k \sum_{i=-M}^{M} b_k(i) s_k(t - iT - \tau_k) \tag{2}$$

where $2M + 1$ is the number of data symbols per user per frame, $T$ is the symbol interval, $A_k$, $\tau_k$, $b_k(i)$ and $s_k(t)$, respectively, the received amplitude, the delay, symbol and spreading sequence of the $k$th user. The spreading sequences are of the form

$$s_k(t) = \sum_{j=0}^{N-1} \beta_j^k \psi(t - jT_c) \tag{3}$$

where $N$ is the spreading gain, $\beta_j^k \in \{-1, +1\}$ and $\psi$ is a normalized chip waveform of duration $T_c$, where $NT_c = T$. Without loss of generality, let us simplify further our model by considering a synchronous system, in which $\tau_k = 0, k = 1 \ldots K$. It is then sufficient to consider the received signal during one symbol interval. The received signal model yields

$$r(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + \sigma n(t), \qquad t \in [0, T] \tag{4}$$

At the receiver, chip-matched filtering followed by chip rate sampling yields an $N$-vector of output samples within a symbol interval $T$

$$\mathbf{r} = \sum_{k=1}^{K} A_k b_k \mathbf{s}_k + \sigma \mathbf{n} \tag{5}$$

where

$$\mathbf{s}_k = \frac{1}{\sqrt{N}} [\beta_0^k \ \beta_1^k \ \ldots \ \beta_{N-1}^k] \tag{6}$$

## 3. SUBSPACE DECOMPOSITION AND LINEAR MULTIUSER DETECTORS

Let us denote $\mathbf{S} \triangleq [\mathbf{s}_1 \ldots \mathbf{s}_K]$ and $\mathbf{A} \triangleq \mathrm{diag}(A_1^2, \ldots, A_K^2)$. The autocorrelation matrix of the received signal $\mathbf{r}$ is then given by

$$\mathbf{C} \triangleq E\left\{\mathbf{r}\mathbf{r}^T\right\} = \sum_{k=1}^{K} A_k^2 \mathbf{s}_k \mathbf{s}_k^T + \sigma^2 \mathbf{I}_N = \mathbf{S}\mathbf{A}\mathbf{S}^T + \sigma^2 \mathbf{I}_N \tag{7}$$

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By performing the eigendecomposition of the matrix $\mathbf{C}$, we get

$$\mathbf{C} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T = [\mathbf{U}_s \quad \mathbf{U}_n] \begin{bmatrix} \mathbf{\Lambda}_s & \\ & \mathbf{\Lambda}_n \end{bmatrix} \begin{bmatrix} \mathbf{U}_s^T \\ \mathbf{U}_n^T \end{bmatrix} \tag{8}$$

where $\mathbf{\Lambda}_s = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$ contains the $K$ largest eigenvalues in descending order, and $\mathbf{U}_s = [\mathbf{u}_1 \ldots \mathbf{u}_K]$ the corresponding eigenvectors; $\mathbf{\Lambda}_n = \sigma^2 \mathbf{I}_{N-K}$ , the multiply defined noise eigenvalue, and $\mathbf{U}_n = [\mathbf{u}_{K+1} \ldots \mathbf{u}_N]$ contains the $N - K$ orthonormal eigenvectors that span the *noise subspace*.

### 3.1. Decorrelating Detector

Based on this subspace decomposition we can reformulate [2] the (non-trivial) linear multiuser detectors. A linear multiuser detector for demodulating the $k$-th user data bit in (5) is in the form of a correlator followed by a hard limiter

$$\hat{b}_k = sgn\left(\mathbf{w}_k^T \mathbf{r}\right) \tag{9}$$

where $\mathbf{w}_k \in \mathcal{R}^N$. The decorrelating detector is the unique signal $\mathbf{d} \in \mathrm{range}(\mathbf{U}_s)$ that minimizes

$$\psi(\mathbf{d}) \triangleq E\left\{\left[\mathbf{w}^T\left(\sum_{k=1}^{K} A_k b_k \mathbf{s}_k\right)\right]^2\right\} \tag{10}$$

subject to the constraint $\mathbf{w}^T\mathbf{s}_1 = 1$. Some algebra over this argument along with the aid of the subspace decomposition and yields the expression of the decorrelating detector in function of the *signal subspace* parameters for the user of interest

$$\mathbf{w}_1^{\mathrm{dec}} = \frac{\mathbf{U}_s(\mathbf{\Lambda}_s - \sigma^2\mathbf{I}_K)^{-1}(\mathbf{U}_s^T\mathbf{s}_1)}{(\mathbf{s}_1^T\mathbf{U}_s)\left(\mathbf{\Lambda}_s - \sigma^2\mathbf{I}_K\right)^{-1}(\mathbf{U}_s^T\mathbf{s}_1)} \tag{11}$$

### 3.2. MMSE Detector

In an analogous form, the Minimum Mean Square Error detector (MMSE) results from the minimization of the MSE, defined by

$$\mathrm{MSE}(\mathbf{w}_1) \triangleq E\left\{(A_1 b_1 - \mathbf{w}_1^T\mathbf{r})^2\right\} \tag{12}$$

also subject to the constraint $\mathbf{w}_1^T\mathbf{s}_1 = 1$. Its expression in terms of signal subspace parameters is

$$\mathbf{w}_1^{\mathrm{MMSE}} = \frac{\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{U}_s^T\mathbf{s}_1}{\mathbf{s}_1^T\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{U}_s^T\mathbf{s}_1} \tag{13}$$

As previously argued [2], both detectors, in terms of the signal subspace can be estimated from the received signal only with the prior knowledge of the signature waveform and timing of the user of interest, thus obtaining them blindly.

### 3.3. Signal to Interference Ratio

A measure of how good a detector is performing is the so called Signal to Interference Ratio (SIR). In terms of the signal subspace this measure can be expressed as:

$$\mathrm{SIR} = \frac{(\mathbf{w}_1^T\mathbf{s}_1)^2 A_1^2}{\mathbf{w}_1^T\mathbf{w}_1\sigma^2 + \sum_{k=2}^{K}(\mathbf{w}_1^T\mathbf{s}_k)^2 A_k^2} \tag{14}$$

where $\mathbf{w}_1^{\mathrm{dec}}$ or $\mathbf{w}_1^{\mathrm{MMSE}}$ is used to obtain the respective SIRs.

## 4. FAST SUBSPACE DECOMPOSITION

Previous attempts to formulate low-complexity subspace-based multiuser detectors have made use of techniques that are rather expensive in terms of floating point operations. We propose to profit from robust and fast algorithms that have been developed with this purpose in mind. This methods, called Krylov Subspace (KS) methods [5][6], have been constructed as "black-box" algorithms whith the sole purpose of solving linear systems such as $\mathbf{Ax} = \mathbf{b}$. The Fast Subspace Decomposition [3][4] uses one particular KS method called the Lanczos algorithm that exploits the structure of the correlation matrix $\mathbf{C}$, i.e. the simmetry and positive-definiteness of $\mathbf{C}$. The Rayleigh-Ritz procedure is used to extract *approximate* eigenvalues of $\mathbf{C}$. These Rayleigh-Ritz eigenvalues are asymptotically equivalent to the actual eigenvalues [3][5].

### 4.1. The Rayleigh-Ritz approximation and the Lanczos algorithm

The Lanczos algorithm is as follows

Given $\mathbf{C}$ Symmetric;
$\mathbf{q}_1 = \mathbf{f}$ (unit − norm)
$\beta_1 = 0$
while $\beta_j \neq 0$
  $\mathbf{z} = \mathbf{C}\mathbf{q}_j$
  $\alpha_j = \mathbf{q}_j^T\mathbf{z}$
  $\mathbf{z} = \mathbf{z} - \alpha_j\mathbf{q}_j - \beta_{j-1}\mathbf{q}_{j-1}$
  $\beta_j = ||\mathbf{z}||^2$
  $\mathbf{q}_{j+1} = \mathbf{z}/\beta_j$
  Compute eigenvalues and eigenvectors
end

The Lanczos algorithm is iterative and runs for $m$ steps; it is used to construct a matrix $\mathbf{T}_m$ from $\mathbf{C}$ with simple operations. The construction of an orthonormal basis $\mathbf{q}_1, \ldots, \mathbf{q}_m$ is the goal of the Lanczos algorithm. This basis forms a set of orthonormal basis for the Krylov subspace defined to be $\mathcal{K}^m \triangleq \text{span}\left\{\mathbf{f}, \mathbf{C}\mathbf{f}, \ldots, \mathbf{C}^{m-1}\mathbf{f}\right\}$. With this basis, the matrix $\mathbf{Q}_m = [\mathbf{q}_1 \quad \ldots \quad \mathbf{q}_m]$ can be used to form

$$\mathbf{Q}_m^H \hat{\mathbf{C}} \mathbf{Q}_m \triangleq \mathbf{T}_m = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \beta_{m-2} & \alpha_{m-1} & \beta_{m-1} \\ & & & \beta_{m-1} & \alpha_m \end{bmatrix}$$

This $\mathbf{T}_m$ is real and tridiagonal and needs not to be constructed by the multiplication of the matrices $\mathbf{Q}_m$ and $\mathbf{C}$ as the $\{\alpha_i, \beta_i\}_{i=1}^m$ quantities are obtained in each step of the Lanczos algorithm, further reducing the complexity of the algorithm. The eigenvalues of $\mathbf{T}_m$, $\theta_i^{(m)}$ are quite easily extracted due to its structure. These are the *Rayleigh-Ritz* (RR) eigenvalues that are used to approximate the eigenvalues of $\mathbf{C}$.

### 4.2. Estimation of the number of signals

The FSD algorithm completes the Lanczos algorithm by using the RR eigenvalues in each Lanczos step to compute a low-complexity statistic to determine the number of signals. This statistic is

$$\varphi_{\hat{d}} = N(M-\hat{d}) \log \left\{ \frac{\sqrt{\frac{1}{M-\hat{d}} \left( \|\hat{\mathbf{C}}\|^2 - \sum_{k=1}^{\hat{d}} (\theta_k^{(m)})^2 \right)}}{\frac{1}{M-\hat{d}} \left( \text{Tr}(\hat{\mathbf{C}}) - \sum_{k=1}^{\hat{d}} (\theta_k^{(m)})^2 \right)} \right\} \quad (15)$$

where $\hat{d} = 0, 1, \ldots, m-2$. Under the hypothesis that the signal subspace is $\hat{d}$ dimensional, it can be shown that $\varphi_{\hat{d}}$ is asymptotically $\chi^2$ distributed with $(1/2)(M-d)(M-d+1)$ degrees of freedom if $\mathbf{C}$ is real and with $(M-d)^2 - 1$ degrees of freedom if $\mathbf{C}$ is complex. Thus, the algorithm to make a strongly consistent decision about the number of signals is to compute the statistic $\varphi_{\hat{d}}$, and accept $d$ as the number of signals if $\varphi_{\hat{d}} < \gamma c(N)$. Here, $\gamma$ is the value where the $\chi^2$ cumulative distribution function reaches a high value, say 0.99 and the function $c(N)$ must satisfy that

$$\lim_{N\to\infty} c(N)/N = 0 \qquad \lim_{N\to\infty} c(N)/\log\log N = \infty$$

say $c(N) = \sqrt{\log N}$. The computation of $\varphi$ takes no additional complexity as it can be calculated in advance and implemented as a look-up table.

### 4.3. Stability and Complexity of the Lanczos Algorithm

The stability of the iterations of the Lanczos algorithm is complete in an infinite precision environment. With round-off errors the Paige's Theorem applies [5]. Given the following eigendecomposition

$$\mathbf{T}_m = \mathbf{V}^T \mathbf{\Lambda} \mathbf{V}$$

if we extract the eigenvectors of $\mathbf{C}$ $y_{k,i} = \mathbf{Q}_k v_i$ we can construct

$$\mathbf{y}_{k,i}^T \mathbf{q}_{k+1} = \frac{O\left(\varepsilon \|\mathbf{C}\|\right)}{\beta_k |\mathbf{v}_i(k)|}$$

This means that the component $\mathbf{y}_{k,i}^T \mathbf{q}_{k+1}$ of the computed Lanczos $\mathbf{q}_{k+1}$ in the direction of the RR vector $\mathbf{y}_{k,i} = \mathbf{Q}_k \mathbf{v}_i$ is proportional to the reciprocal of $\beta_k |\mathbf{v}_i(k)|$, which is the error bound on the corresponding RR eigenvalue $\theta_i$. Thus, when $\theta_i$ converges and its error bound goes to zero, the Lanczos vector $\mathbf{q}_{k+1}$ acquires a large component in the direction of the RR vector $\mathbf{y}_{k,i}$. Thus the RR vectors become linearly dependent. This can be corrected by a Gram-Schmidt procedure in each Lanczos iteration that renormalizes the Lanczos vectors as the RR vectors converge. This procedure increases the complexity of the Lanczos algorithm from $O(KN)$ to $O(K^2N)$. This can be corrected by computing the errors of the converging eigenvalues and orthogonalizing only the eigenvectors that need it. This results in a marginal increase in the complexity in the algorithm. This is called the Selective Orthogonalization (SO) Lanczos.

## 5. RESULTS

We have implemented a *Blind Adaptive Krylov Subspace Multiuser Detector* (BAKS) with a SO Lanczos algorithm in its core. The users in the CDMA system have been assigned Gold codes of 31 chips. In the figure 1 the convergence of the SO Lanczos algorithm can be checked. $K = 30$ users have been used for this test. It can be seen that the convergence is sure at 32 Lanczos steps. The actual eigenvalues can be compared at the end of the Lanczos iterations. They are represented as circles. The selected eigenvectors for re-orthogonalization can be seen for the same example in figure 2. As the Lanczos procedures evolves, the eigenvalues in both extremes of the spectrum are the first to converge (see figure 1(a)). Thus, the corresponding eigenvectors are the firsts to lose orthogonality, thus needing reorthogonalization. This can be seen in figure 2. As the whole spectrum converges to its actual eigenvalues, the corresponding eigenvectors in the next Lanczos steps need reorthogonalization. This can also be observed and supporting the hypothesis that further Lanczos steps that $K+2$ are not needed to assure convergence. Furthermore, the number of eigenvectors being reorthogonalized to convergence are but a fraction of the total computed Lanczos eigenvectors.

In the following experiment, a simulation of 6 CDMA users is done. We have a SNR for the user of interest after despreading of 20 $dB$ and $A_k^2/A_1^2$ uniformly distributed in the interval $[0, 30]\,dB$ for $k = 2, \ldots, 30$. A total of 8000 samples are considered. At $t = 3000$, two users exit the channel and at $t = 6000$, they reenter. The BAKS-MMSE algorithm is checked against the PASTd-MMSE algorithm introduced in [2]. The PASTd-MMSE algorithm is fed with the results of a SVD of the signal subspace during the first 100 samples, each time the conditions change, i.e. $t = 0$, $t = 3000$ and $t = 6000$. The PASTd-MMSE is equally fed with the exact number of users actually present in the channel. These measures are taken in order to obtain a minimally competitive behavior form the PASTd-MMSE algorithm. Figure 3 represents the SIR for the user of interest and for the BAKS-MMSE detector. The data plotted here are the average over 20 simulations. The performance of the Matched Filter is plotted for reference. Although the BAKS-MMSE continually estimates the number of users and needs not to be initialized with "good" estimates, both its convergence and its dynamical behaviour are well beyond reach of the PASTd-MMSE algorithm.

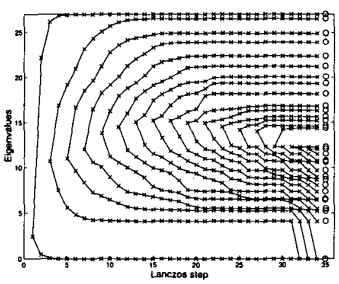


**Fig. 1.** K=30 and N=127 (× computed, ○ actual)

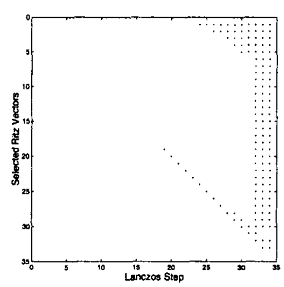


**Fig. 2.** Selected eigenvectors for reorthogonalization

## 6. CONCLUSIONS

We have constructed a low-complexity blind adaptive multiuser detector that outperforms previously multiuser detectors [2] both in static and dynamic channel conditions. As the PASTd-MMSE has been shown to be superior to the Minimum Output Energy (MOE) [2] multiuser detector, this shows that the BAKS-MMSE detector is preferable to other detectors for its implementation in actual receivers, i.e. UMTS-TDD receivers.

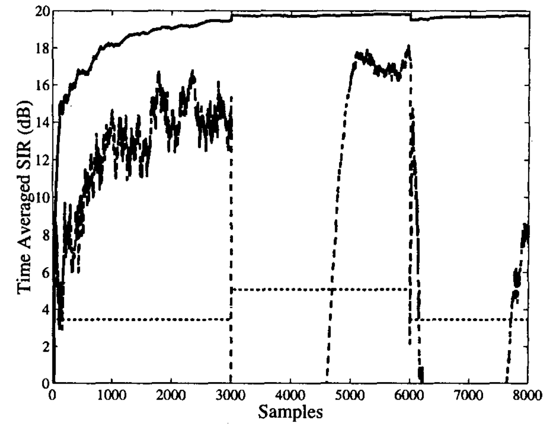


**Fig. 3.** Time Averaged SIR for the user of interest. The continuous line represents the results of the BAKS-MMSE algorithm, the dashed line, that of the PASTd-MMSE and the dotted line is the output SIR of the Matched Filter.

## 7. REFERENCES

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