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TESIS DOCTORAL

Uncertainty and Complex Dynamics in Econophysics

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A Loreto y May, con amor.

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Summary

"Not all those who wander are lost"
-J.R.R. Tolkien, *The Lord of the Rings*

During the recent history of humanity, on the one hand economic systems of advanced civilizations have achieved a great capacity in supplying their citizens with unimaginable wellness, but on the other hand their devastating power of destraction has made us doubt about the fragile security that we think they grant us.

The wounds left after the last financial crisis of 2008 are not healed yet, but they serve us as a reminder that unlike what many people believe or want to believe, this economic beast is not ready to remain immobile, trapped in an equilibrium state for the rest of its life. However, these tragic events sometimes produce positive outcomes, they force us to return to the designing table and reconsider our models, paradigms, theories and even our knowledge.

I was in charge of a small familiar business when the crisis hit Spain. I felt on my own flesh the huge social and economic recession that was generated by this catastrophic event. My curiosity and anxiety for knowing, drove me to question everything I knew or thought, borrowing ideas from a variety of scientific fields that in principle do not have anything to do with economics. This thesis is the result of this long search for the truth.

This thesis has been developed during my years, researching in the Nonlinear Dynamics, Chaos Theory and Complex Systems research group of the Universidad Rey Juan Carlos. In this thesis we develop and study nonlinear models in economics capable of producing so complex dynamics, that our own understanding of predictability is challenged. Next, I will briefly describe the structure of this work.

Chapter 1. Introduction

In this chapter we will build the pillars of this work. We will describe how standard economics theory has been challenged by the emerging science of complexity. This new scientific paradigm has changed the way economists look and understand

the entire economic system and it also opened the door for physical scientists to apply their own tools and methods to study economic phenomena, giving way to the novel econophysics approach. In this work, we have used it broadly, for this reason, concepts like complex systems, interdisciplinarity, chaos, basins of attraction, fractals, agent based models (ABM) and emergence are briefly introduced. These ideas will recurrently come up along the whole thesis.

Chapter 2. The supply based on demand method

In this chapter we introduce the supply based on demand dynamical model (SBOD). This model is one of the props of this thesis and we invoke it in Chapter 3. For this reason, we describe very carefully the SBOD model. We explain the reasoning behind it, and the motivations that led us to develop this model in the first place. Afterwards we describe the structure of the model and how it works. Finally, we study in detail the complex market dynamics produced by this model and the important observation that *the last bifurcation means market collapse.*

Chapter 3. Preventing the crash with partial control

The partial control method was developed by our research group with the collaboration of Prof. James Yorke from the University of Maryland. This novel control method has been successfully applied to a variety of systems in science and engineenering achieving remarkable results. Here we apply for the first time the partial control method in an economic context, applying it on the supply based on demand model introduced in Chapter 2. We demonstrate that the firm is capable of controlling the market from the bottom up, applying much smaller control than the market noise.

Chapter 4. When repetition is the best strategy

This chapter is devoted to the study of the system proposed by the Complexity Challenge Team from the Santa Fe Institute at Spring 2018. This intriguing problem is an extension of the original *El Farol Bar problem* in which more complexities are introduced. There are three pools and each one of them has different rewarding schemes that depend on the attendance to the pool and in some stochastic functions. Each agent must locate itself at each time step in one of the pools with the goal of maximizing its total balance, which is the reward paid by the pool minus some cost. To study this problem, we have developed an agent based model that integrates 13 types of strategies.

Chapter 5. Making predictions on fractal basins

There are situations where our idea of deterministic predictability is challenged by the mere existence of complex structures on the phase space of some dynamical systems. When these basins of attraction present fractal structure, the predictability of any event in these fractal regions depends on the accuracy in which we measure the initial conditions, and this is a technical problem. In this chapter, we study the relationship between the global predictability and the local measure of uncertainty in a dynamical system that presents the Wada property. We demonstrate that the probability of ending up in each basin of attraction converges to a fixed probability when we increase the accuracy of the measurement or the phase space resolution. This means that globally, the probabilities of each final state of the system are fixed, although locally we will never be able of predicting the orbits.

Chapter 6. Results and Discussion

This chapter is devoted to the description of the main results of this thesis. We also propose some possible research lines for a future work.

Chapter 7. Conclusions

The main conclusions of this thesis are summarized in this chapter.

Contents

Chapter 1
Introduction

"We are a way for the cosmos to know itself"
-Carl Sagan

1.1 The dark energy behind our expanding economies

If there is something that all economists agree on, is that the economy is complex - very complex. Approximately 250 years ago, one of the fathers of the economic thought, Adam Smith, in his famous book *The Wealth of Nations*, introduced the notion of the *invisible hand* to illustrate the mysterious force that somehow enforces the markets to work effectively. However, he could only explain this discovery with words, since the necessary tools needed to generate a concise theory that explains quantitatively this phenomenon, did not exist in those days.

The neoclassical economic theorists that came after Smith, took another direction in the study of this and other economic phenomena. Inspired by the efficiency of the exact sciences to explain physical and chemical phenomena, these economists mathematize the economy using mathematical tools that did not exist in Smith days. These economists influenced by the theory of thermodynamics [1, 2], developed the demand and supply model, which serves as one of the main paradigms in modern economics. This theory has supported many of the practical and theoretical ideas that were very successful in explaining and supporting the economic growth experienced during the 19th and 20th centuries that was mainly based on production.

The mathematical arsenal that these economists used to create their new theories and models, is very powerful in explaining phenomena with few variables, for example the business cycle [3], strategy game models [4, 5] or the market equilibrium [6]. Other mathematical weapons from this arsenal are very precise in studying phenomena with thousands of millions of variables exploiting the different laws of statistics, for example the famous efficient market hypothesis studied by Louis Bachelier and Paul Samuelson [7, 8, 9] or Black and Scholes options valuation formula [10]. But when we are dealing with phenomena that are characterized by having a considerable number of variables that is neither very small nor extraordinarily big, this mathematical artillery is slightly effective. And exactly in this place in the spectrum of phenomena, is where scientists think that many of the most complex economic phenomena like value or market organization lays [11, 12].

Approximately 200 years after Adam Smith published his ideas about the invisible hand, the Nobel prize economist Friedrich von Hayek, replace the term *invisible hand* by another extreme idea: *the spontaneous order of the market*. Von Hayek held that the market arises spontaneously from the social interactions and from the human actions. He also claimed that the basic structure of the free markets guarantees the efficient assignment of the economic resources, better than any system designed by man. For this reason, he was convinced that these phenomena cannot be understood by any of the methods of the natural sciences. Additionally, he argued that it is not necessary to interfere in any way in this spontaneous order imposing laws or behaviors that may disturb the natural evolution and dynamics of the markets.

The invisible hand or the spontaneous order of the market are just two powerful ideas that try to explain the mysterious force that drives our economy - a force that we can only deduce from the interactions between economic observables such as the demand the supply or the price, but we cannot measure it directly. Exactly like the dark energy - the mysterious energy responsible of the expansion of our universe. Almost all scientists agree that is there, but no one understand its true nature, yet.

1.2 A paradigm shift

It is more than clear that our knowledge depends heavily on the tools that we have in our disposal to explore the nature that surround us. The invention of telescopes helps us to discover the real immensity of our universe and the physical laws that govern it. Microscopes allowed us to observe the complexity and the mysteries that the microscopic world was keeping away from our naked eyes. And advance computers increased dramatically our computing power enabling us to explore the huge complexity generated by a simple interaction between bouncing balls [13], the way in which populations grow [14], to make better weather predictions or even make use of algorithms that mimic our own brain. Our technology allows us to feel what our senses cannot feel, to explore places where our feet cannot take us and to know what our reason is unable of deducing.

From the last half of the 20th century, we have witnessed a huge boost in computation technology. This technological revolution has changed our lives considerably. But it also changed the way we face very complex problems. Computational power harnessed to solve complex scientific problems by solving millions of operations in a blink of an eye, gave scientists one of the keys to the nonlinear kingdom, allowing them to find numerical solutions to nonlinear equations or to systems of nonlinear equations in few seconds. If in the past, economists tried to avoid nonlinearities by simplifying considerably their equations, sometimes with the cost of losing all the richness presented in the phenomenon they try to model. Today they can study complex systems upside-down from the bottom up, starting by modelling the agent's behavior and then simulate how these behaviors aggregate into the global behavior of the system.

This new tool allows economists for the first time in history to create imperfect worlds, with imperfect agents (heterogeneous and with bounded rationality) that interact within themselves in an imperfect environment. A world more similar to the world in which we live. In this way we can contemplate how the economy - is not deduced but emerges. From the extensive literature on complexity in economics, I want to highlight the work of Joshua M. Epstein and Robert Axtell on Sugarscape - evolution of artificial societies [15], the search of optimal strategies in iterative strategic games by Robert Axelrod in his famous tournaments [16, 17], Nobel Prize Thomas Schelling's fabulous study on focal points and economic interactions [18, 19, 20] and the *El Farol bar problem* by Brian Arthur and his incredible study about technology [21, 22].

The neoclassical economists have assumed that macroeconomics phenomena could be deduced from the microeconomic system, in other words, the rational behavior of the microeconomic agents is added up to the linear dynamics predicted by the macroeconomic theory. But the absence of evidence of these rational agents [21, 23, 24, 25] or such linear dynamics in the economy brought a lot of criticism upon this approach. Empirical data show that the devastating economic crises among other phenomena that strike our modern economies [26], do not match with the neoclassical view of the economy. And at this point the complexity economics paradigm with its powerful computational simulations is taking the relay [27, 28].

Complexity economics observes the same economy, but from another angle - using another paradigm - nonlinear models, describing the economy as an open system, composed by heterogeneous agents with bounded rationality, making decisions in very specific contexts, that give place to networks of interactions that form institutions. This economy is immersed in a very complex dynamics out of equilibrium, constantly evolving and adapting to the new environment. In this thesis we study the economy from this point of view - from the complexity paradigm. Shedding more light on our understanding of our economy.

1.3 Scientific context

Once the general framework of this thesis has been set, we will introduce briefly some of the scientific concepts that will be accompanying us along all this work.

1.3.1 Complex Systems, interdisciplinarity and holism

The exact definition of a complex system does not exist yet. Throughout the last decades with the emergence of this new science, many scientists have tried to clarify what is a complex system and how to measure the intrinsic complexity in such systems [29, 30, 31]. But there is no consensus among scientists about the exact definition nor how to measure the complexity of these type of systems. For this reason, instead of exposing here the different definitions of a complex system I will briefly introduce some of their most important properties.

- Simple components - these systems are constituted by very simple components.
- Information - the components that constitute these systems create, use, transmit and compute information.
- Nonlinear Dynamics - these systems constantly change their structure and behavior in a nonlinear fashion. Also, as a result of nonlinear interactions between their components.
- Evolution and learning - these systems tend to adapt to new situations and environments, learning and changing over time.
- Absence of a central control - there is no one who directs this orchestra.

Now, after considering these key properties, it should be obvious that in order of studying and understanding this type of systems, we must integrate many ideas from a variety of scientific fields (physics, biology, mathematics, etc). We call Interdisciplinarity to this integration of ideas and the usage of models and theories to explain different phenomena far from the scientific field in which they borned. Additionally, we must change our study strategy, from reductionism to holism taking as granted that *the whole is larger than the sum of its parts.* As Péter Érdi quoted "The science of complexity suggests that while life is in accordance with the laws of physics, physics cannot predict life" [32].

The markets are a classic example of a complex system and in this thesis, we will study the markets extensively. In the next chapter we introduce the *supply based on demand dynamical model* (SBOD), a very simple model capable of producing very complex market dynamics. In Chapter 3 we show how to control the market preventing a market collapse. In Chapter 4 we study a problem related to a complex market desicion using an agent based model.

1.3.2 Chaos Theory and Chaos in Economics

The first and most important observation that led humanity to familiarize with chaotic phenomena, is that everything is in motion. A long before Heraclitus said *panta rei*, Debora sang: "The mountains flowed before the Lord" because she knew that, in God's infinite perspective, everything seems flowing [32].

But only at the end of the 19th century and after approximately 200 years in which most of the scientists believed, in the Newtonian universe - universe like a clock, the French mathematician Henri Poincaré, after studying the three-body problem, would offer us the first mathematical evidence of chaos [33]. Approximately 70 years afterwards, the American meteorologist Edward Lorenz, using his computer, discovered a very strange phenomenon, that he would popularize as the butterfly effect [34]. In 1975, 12 years after Lorenz discovery was published, James A. Yorke and Tien-Yien Li, two American mathematicians, published a paper titled *Period three implies chaos* in which they called this phenomenon for the first time - Chaos [35]. Chaos means that tiny variations in the initial conditions can lead to enormous deviations in the orbits of the system, making predictions almost impossible. Chaotic systems present strong sensitivity to the initial conditions.

The first studies of chaotic phenomena in economics appeared at the beginning of the 80's of the 20th century with the works of Jess Benhabib and Richard H. Day. Their work showed to the scientific community, the importance and the potential of applying chaos theory in economics. Since then, chaos theory has been applied in many models in economics, growth models, competition models, markets models, fiscal models and many more [1].

1.3.3 Fractal Geometry

If you have observed the structure of a coastal line of some country from a plane, you could convince yourself that the coastal line does not follow a straight line, but rather a more complex structure. If you have approached more closely, observing the same coast line from a cliff for example, you would see the same pattern. If you get even more close, standing on the coastline, you would continue seeing the same pattern in the stones and rocks that form this coastline. The mathematician Benoit Mandelbrot was the first one who manage to formulate this observation in a new type of geometry that he called *fractal geometry* [36].

He wrote in his famous book *The fractal geometry of nature*, "Why is geometry often described as cold and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." [37]. Curiously, Mandelbrot was very interested in the shape of price charts of different assets in different markets and he was able to show that price charts can be described by fractals and multifractals [38, 39].

The fractal geometry beside of generalizing the Euclidean geometry, describes more faithfully the shape of nature. Technically, fractals are defined as geometrical objects with a Hausdorff-Besicovitch dimension larger than their topological dimension. These objects are self-similar in all scales and this is one of the causes of uncertainty in nonlinear systems. We will learn that this property can be very useful in some cases when we want to make predictions.

1.3.4 Basins of attraction, Wada Basins and predictability

In dissipative systems, many final states are possible. The set of initial conditions that lead to a certain final state of the system, is called a basin of attraction. One of the most intriguing features of these basins of attraction, is their geometrical structure. When there are two or more basins of attraction over the phase space of some dynamical systems, the basin boundary that separates the different basins might be a smooth curve or can be instead a fractal curve. When the basin boundary is fractal, the system is much more difficult to predict. There is an extreme case of fractal boundary, called the Wada boundary, in which three or more basins share the same fractal boundary.

In this situation we say that the system possesses an extreme sensitivity to the initial conditions, since the final state of the system will depend on the accuracy in which we measure the initial conditions. Minuscule differences in the measurement of the initial conditions inside the Wada boundary will be translated in very different final states of the system. Chapter 5 is devoted to the study of the Duffing oscillator when it possesses this type of basins.

1.3.5 Agent Based Modeling

Paraphrasing Galileo Galilei from "Il Saggiatore" *...the universe is written in mathematical language*. There is no doubt that mathematics is a very useful language to model natural phenomena. But sometimes it is very hard to reduce the behavior of some natural or social phenomena to a series of equations. In these situations, we can use another type of modeling techniques that take advantage of computers capacity to execute millions of repetitive computations in a very short time.

One of these modeling techniques is called Agent Based Modeling (ABM), and it evolved throughout the 20th century along with computation. This technology was developed by many influential scientists like John von Newman, Stephen Wolfram, John Conway, Robert Axtell, John Holland, Joshua Epstein and Uri Wilensky among many others. This type of computational models aims to simulate the actions, behaviors and the interactions of autonomous agents or agent's collective entities to explore the impact that an agent or an agent behavior have on the system as whole. Unlike cellular automata (CA), ABM's have the potential of studying how a variant in the agent properties like the number and kind of agents presented in the system or some particular agent behavior affects the dynamics of the system. The agents behave independently, but they interact with the environment and with other agents that surrounds them. In this way we can contemplate how some unexpected global dynamics emerge from these interactions that we could not appreciate otherwise [40].

We will dedicate Chapter 4 to study a complex decision problem proposed by the Santa Fe Institute's Complexity Challenge Team in Spring 2018. We study this intriguing problem from two directions, the ecosystem of possible strategies and the global dynamics that emerges from the interactions between the different agents using an agent based model that we have developed especially for this challenge.

1.3.6 Emergence

An emergent phenomenon [41] appears when the collective behavior of the agents that compose the system can not be deduced by only reducing the system to its constituent elements. In physics the phenomenon of superconductivity is understood as an emergent phenomenon. But in the life sciences these phenomena are very common. Perception, thought, feelings or consciousness cannot be understood by reducing the basic components of our brain- the neurons. Life itself cannot be understood by reducing the chemical components of organic matter and societies cannot be understood as the simple sum of their citizens. In each one of these systems the global behavior, what we call, superconductivity, consciousness, life, societies and also economies are emergent phenomena.

Chapter 2

The supply based on demand method

"Supply and demand constantly determine the prices of commodities; never balance, or only coincidentally; but the cost of production, for its part, determines the oscillations of supply and demand.

-Karl Marx

As stated in the introduction, the standard economics modeling approach assumes that the economy is a close system where two dynamics are possible: explosion or equilibrium. Once we get rid of this assumption by modeling the market in a nonlinear fashion, we observe many additional dynamics that in some cases remind us real price charts. One of the main assumptions made in the demand and supply model, is that the quantity demanded, and the quantity supplied in a market, are two independent variables. The only way of influencing the shape or the position of the demand or the supply curves, is by applying an external force to the system. It is very common to use the technology as an example, of an exogenous force capable of moving the supply curve from its original position. But in the real world one of the most important factors that determine the quantity of supply is the quantity demanded or rather the estimated demand in the market. Many firms from all the scales, invest many resources to study and estimate the future demand, since this information might be crucial for their future survival. Firms try to optimize their production in a situation of uncertainty of demand by estimating the demand.

In the literature this problem is known as *the newsvendor problem* and it dates back to the end of the 19th century. Since then and up to today many articles that analyze mathematically and statistically this problem have been published, but very few works focus on the effect of this firm behavior on the market dynamics. In this chapter we develop and study a simple dynamical model that tries to capture some of the complexities involved in this iterative process of estimating the demand and producing according to it. We show that many market dynamics are possible in this setting and some of them are chaotic.

2.1 Introduction

Firms need to make the decision of how many goods to supply before they even know how many goods the market will demand in the next sales season. This problem is known as *uncertainty of demand forecast* and it has been widely studied in economics and supply chain management [42]. Being successful in predicting the future demand might be crucial for the survival of any productive company in a competitive market. The standard microeconomics models of the firm assume perfect information, implying that the firm knows exactly the shape of the demand curve. Furthermore, these models assume static and independent demand and supply curves, so that the decisions made by the firm do not have any effect on the shape nor the slope of the supply and demand curves. In this position the firm is only maximizing profits and its actions have no influence on the global dynamics of the market. So that in the long run the system settles down in equilibrium. Since the publication of the classic paper of George A. Akerlof [43], new models have been proposed. For example, pricing models of the monopoly under uncertainty of demand, considering the demand as a stochastic function [44, 45, 46], focusing on the optimal price for the firm and less on the market dynamics. Complexity economics in contrast, focuses on the emerging market dynamics created by economic agents when they react to patterns created by their own interactions during the time they interact [47, 40, 28]. This approach focuses on the connectivity and the interdependences between economic agents and how they organize and interact to achieve their economic end. Modeling the economy in this way opens up a new world of possibilities, where equilibrium is one possible dynamics among many others that can emerge from these interactions. In recent years, policy making have adopted Dynamic Stochastic General Equilibrium models (DSGE), to better predict and even control the economy at the macro level [48, 49]. These models are built from three main blocks where each one is a representation of some economic agent or a group of agents. The demand block represents the consumption of households, firms and even the government. The supply block represents the productive agents of the economy and the policy block represents financial institutions like central banks [50]. These kind of models add to the general equilibrium models some simple dynamical interaction between the economical agents in addition to some stochastic external shocks. In supply chain management when the firm has data sets it almost always uses statistical methods like time series analysis or linear regression [51] to estimate the future demand. We find the combination of the last three frameworks interesting in nonaggregable models at the micro level of the economy.

In this chapter, we will think about the market as a dynamical place where one firm is a price maker while it has limited information about the demand. We focus on firms whose commercial activity involves producing or buying some stocks of a certain good with the purpose of selling them to obtain profits. These firms, mainly small, medium or entrepreneurs do not spend much resources in demand forecasting, they rely mainly on their buyers expectations among limited data sets of past sales, for example small stores, retailers or small factories that their main revenue comes from certain holiday tradition. For simplicity, we will call suppliers, to all the agents that belong to this group.

We will study numerically a dynamical model that is built similarly to a DSGE model without the stochastic terms, focusing on the micro level of the economy. This model is highly inspired by the classic cobweb model [52] with the difference that the supplier decision of how many goods to produce is sensitive to the quantity demanded instead of the market price of the good. The following two key concepts in the supply organization are captured in the model. First, as in the nonlinear version of the cobweb model, we present one possible dynamical procedure based on suppliers expectations [53, 54, 55, 56], that can lead to market equilibrium and to chaos as well. The second idea embodied in this chapter, is that for a given quantity of supply the supplier fix some price that generates a demand feedback from the market. This information is needed to compute the quantity of supply in the next time step. As in real markets the supplier reacts to these demand feedbacks, what creates a rich price-quantity dynamics. Additionally, we will show that in some cases the supplier may push the market towards an equilibrium motivated by his selfish interests, (sell all the stock), as Adam Smith once wrote: *"It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own self-interest..."*. But in other cases the supplier may produce irregular dynamics that may lead to market collapse. We have found that the price elasticity of demand (PED) and the gross margin can play an important role in the stabilization of prices in the same way they can make the market crash.

The structure of this chapter is as follows. Section 2.2 is devoted to the description of the supply based on demand model. Two types of suppliers and their behaviors are described in Section 2.3. In Section 2.4 we introduce the methodology. The global dynamics and results are described in section 2.5. In Section 2.6 we emphasize the idea that the final bifurcation means - market collapse. We describe the influence of the price elasticity of demand (PED) on the global dynamics in Section 2.7. Finally, some conclusions are drawn in Section 2.8.

2.2 Model description

We consider a supply and demand model of the form,

$$D_{n+1} = a - bP_{n+1}, \tag{2.1}$$

$$S_{n+1} = D^{Exp}_{n+1}, \tag{2.2}$$

$$P_{n+1} = \frac{ATC}{1-M}, \tag{2.3}$$

where the quantities demanded and supplied, D_{n+1}, and S_{n+1}, and the price, P_{n+1} are assumed to be discrete functions of time. The parameters a and b are positive constants $a, b \geq 0$ and D^{Exp}_{n+1}, is a funcional of the expected demand. The parameter M, is the gross margin added by the supplier to obtain profits, where $0 \leq M < 1$ and ATC is a functional of the average total cost function of the good, that we will explain in details later on.

The quantity demanded in the market depends mainly on the price of a given good. The price of the good in contrast, depends heavily on the average total cost function, which is directly linked to the quantity of supply. When the supplier decides how many goods to produce, he always estimates in some way the future quantity of demand, D_{n+1}^{Exp}. The problem is, that the supplier makes the decision of what quantity to supply, S_{n+1}, before he knows the reaction of the market to the price that he fixes. In this model we assume, the supplier does not know anything about the demand function. The only available information he has, is the quantity demanded at the price in which he sold his products in the last sales seasons. We assume an ordinary good market in which, when the price increases, the consumption of the good decreases and vice versa. For simplicity, we assume a linear demand curve with negative slope as shown in Eq. (2.1). Before we proceed, we introduce two more mechanistic assumptions, that describe how the supplier operates in the market.

Assumption 1

The supplier is the only one who sets and adjusts the price in light of circumstances.

In this model the supplier is the only one who sets and adjusts the price. Notice that after the supplier launches the goods into the market, no changes can be done in the quantity supplied nor the price. The price structure is given by the ATC function and the gross margin as shown in Eq. (2.3). Both building blocks are known and controlled by the supplier.

After estimating the demand for the next period, the supplier begins the production phase. He introduces his estimations in the ATC function to obtain his average total costs of production. We assume in this model that the average total costs is computed adding the fix costs to the variable cost per unit of good, divided by the total amount of goods produced. However, there are many possible ways to describe an ATC function. For instance, in many industries the prices lists shown to the buyers are organized in a "piecewise function" fashion, where the price of the good is well established for every subset of quantities the buyer is willing to buy. But here, to stay faithful to the classical cost theory, we have chosen a typical continuous cubic total cost function, that gives rise a parabolic ATC function that depends also on the quantity of production Q [57] as shown in Fig. 3.3. In some markets the average total cost decreases as the supplier increases the amount of goods he produces or the amount of goods he is willing to buy, until reaching some critical point. After crossing this point every additional product produced or bought increments the average total cost. The parabolic shape of the ATC function as shown in Fig. 3.3 captures this idea. In the classic supply and demand model is taken for granted the linear positive slope shape of the supply curve what guarantees convergence towards an static equilibrium. In our case the supply curve is nonlinear, what produces more complex dynamics. The quantity of production Q is the same as the quantity of supply, S_{n+1}, or the expected demand estimated by the supplier earlier, as shown in Eq. (2.2) and (2.4),

$$ATC = \frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2. \tag{2.4}$$

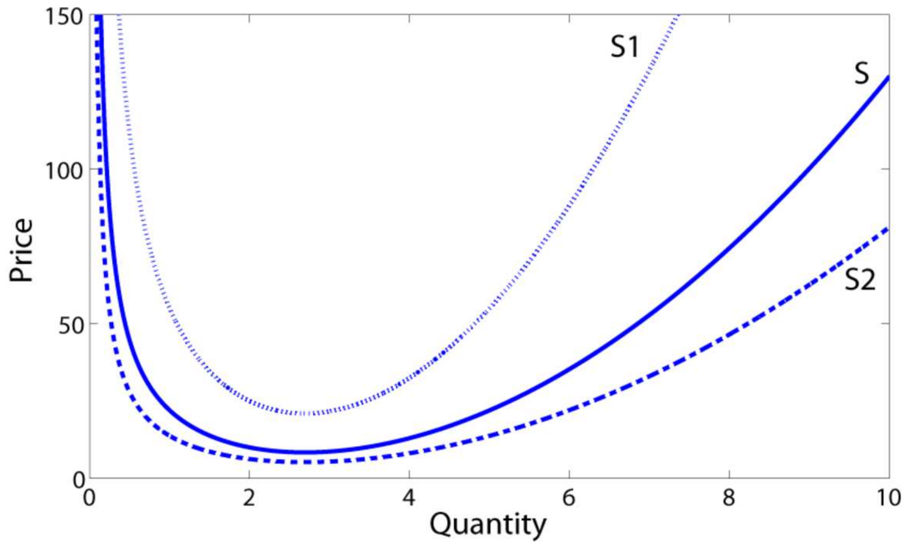

Figure 2.1. Parabolic price function. We have used the following function $P = \frac{1}{1-M} \cdot (F_c Q + v - vQ + Q^2)$, to relate the price of the good with the quantity supplied, where P is the price of the good, Q is the quantity, F_c is the fix cost of production and v is the variable cost of production. Notice that the ATC function inside the brackets, determines the parabolic shape of the price function. The parameters are fixed as: $F_c = 10$ and $v = 4$. The supply curves S as solid line, $S1$ as dot line and $S2$ as dash-dot line, correspond to the gross margin $M = 0.5$, $M = 0.8$, $M = 0.2$ respectively. When the supplier increases the gross margin M, the price of a given quantity of goods increases as well.

We assume that the variable cost, v, and the fix cost, F_c, are positive constants. The final step in this process is to add profits over the average total cost of the good, using the gross margin operator shown in Eq. (2.3). When M increases, the price function moves upwards, what leads to higher prices and when it decreases the price moves downwards what leads to cheaper products as shown in Fig. 2.1.

Assumption 2

The main goal of the supplier is to sell all the produced goods.

For simplicity, we assume that the supplier cannot keep goods as inventories from one period to the next and also he does not maximize his profits. This model does not take into account the financial constraints of the production process, and we assume that the supplier has money to produce or to buy at any point in time. The main focus of the model is to show how the supplier tries to match his expectations about the demand with the real demand in the market and how this process alters the price. So the question is, how the supplier knows if he had a successful sales campaign. In this case, for him, we consider that a successful sales campaign means that all the goods were sold. This is exactly the market equilibrium assumption except that in our model, is just a temporal state of the system and not a constant reality of the market. The supplier quantifies his success after each period using a very simple model - he divides the quantity demanded at time n by the quantity supplied at time n as shown in Eq. (2.5). We call it the *signal of success* (S),

$$S = \frac{D_n}{S_n}. \tag{2.5}$$

According to the signal of success, the supplier decides how many goods to produce and supply in the next period of time. From the mathematical point of view, it is important to notice that the supplier reacts to the signal of success and not implicitly to the quantities demanded and supplied. This simple idea helps us to model the market assuming no inventories and inequalities between demand and supply. The signal of success can be divided in four subsets of outcomes, each one with its corresponding economic meaning. We assume that all outcomes are in the positive domain.
1. When $\frac{D_n}{S_n} = 0$, there is no demand, or even worst, there is no market. In this case the supplier will not produce anything for the next period due to the scarcity of demand.
2. When $0 < \frac{D_n}{S_n} < 1$, the quantity demanded is smaller than the quantity supplied at the given level of price. The supplier produced more goods than what the market could possibly absorb. From the economical point of view, the supplier will probably affront economic losses and also gain negative expectations about the future state of the market.
3. When $\frac{D_n}{S_n} = 1$, the quantity demanded is exactly equal to the quantity supplied. This means that he had a successful sales campaign, exactly as we defined earlier. In general, suppliers aspire to find themselves in this situation. This is a natural equilibrium point of the system as we will show in the following sections.
4. When $\frac{D_n}{S_n} > 1$, the quantity demanded is larger than the quantity supplied. This is a stock-rupture situation. Although the supplier sold all the goods he produced, and this condition meets Assumption 2, losing the possibility to sell even more goods and earn extra revenue, is an unsatisfactory situation for him. Imagine costumers entering through the shop door with money bills in their hands asking for some product that is out of stock. Although he has lost some extra revenue, he gains positive expectations about the future.

The model works as follow, in the first step the supplier supplies some quantity of goods to the market to get some feeling about the demand (seed). Then he observes the quantity of goods that were demanded at the price that he fixed. According to this quantity the supplier decides how many goods to produce or buy for the next period using a simple model that quantify the success of his sales campaign. We called it the signal of success, and it is a simple division between the demanded and supplied quantities at time n. After computing the signal of success the supplier uses it to estimate the expected demand in the next period. The second step is the pricing process. The supplier uses his ATC function to compute the goods average total cost. After obtaining the cost per unit, he adds some profits over the cost using the gross margin operator. Finally, he introduces the goods with their new price into the market. He waits some time until he sees how many goods have been sold and then he repeats all the process again.

2.3 Two simple supplier behaviors

In this Section we will describe two types of suppliers. Both of them share the function that describes the relationship between the signal of success and the multiplier of production for the next period of time, that is, how the amount of goods produced or bought in the present period of time for the coming sales campaign, is affected by the signal of success. In Fig. 2.2 we show this relationship. For the sake of simplicity we have used two very simple suppliers that can be modeled analytically. But in the model, more complex suppliers could be introduced.

The naive supplier

The simplest assumption of all is that the supplier makes the decision of how many goods to supply in the next period, using the signal of success and the amount of goods he supplied in the previous period as a benchmark. The supplier uses a very simple model to compute the expected demand, that works as follows. He multiplies the signal of success with the quantity supplied in the previous period as shown in Eq. (2.6),

$$D_{n+1}^{Exp} = (\frac{D_n}{S_n}) \times S_n = D_n. \tag{2.6}$$

The logic behind this model is that the supplier expects the demand to behave in the next sales season, exactly the same as it behaved in the previous period. This forecasting method is the same as the moving average method with exponential smoothing coefficient of $\alpha = 1$, putting all the weight of the forecast on the most recent information [58]. There is a linear relationship between the signal of success and the multiplier for the next production as shown in Fig. 2.2. The supplier is going to produce exactly the same quantity that was demanded in the previous period. For this reason we have called naive, to this supplier. The model takes the following form

$$D_{n+1} = a - bP_{n+1}, \tag{2.7}$$

$$S_{n+1} = D_n, \tag{2.8}$$

$$P_{n+1} = \frac{1}{1-M} \cdot (\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2). \tag{2.9}$$

Simplifying this system of equations, we get the following one dimensional maps for the demand and the price,

$$D_{n+1}(1-M) = a(1-M) - b(\frac{F_c}{D_n} + v - vD_n + (D_n)^2), \tag{2.10}$$

$$P_{n+1} = \frac{1}{1-M} \cdot (\frac{F_c}{a - b(P_n)} + v - v(a - b(P_n) + (a - b(P_n))^2). \tag{2.11}$$

The cautious and optimistic supplier

This type of supplier is in fact a family of infinite number of suppliers, each one with a different sensitivity to the signal of success. This supplier instead of merely using the signal of success as it is, prefers to transform it to be able to improve the prediction of the demand in the next period. He uses a very simple but powerful model. He finds the nth root of the signal of success where m defines his cautiousness and optimism as we will see next. The supplier multiplies the nth root of the signal of success with the quantity supplied in the previous period that serves him as bench mark. We can see this model in Eq. (2.12),

$$D_{n+1}^{Exp} = \sqrt[m]{(\frac{D_n}{S_n})} \times S_n, \tag{2.12}$$

where $m > 0$. From Fig. 2.2 we can see that when m increases the supplier becomes less optimistic and more cautious about the future state of the market, when the signal of success is greater than one. But he becomes less cautious and more optimistic when the signal of success is between zero and one. This behavior remaind loss aversion [59], where the suppliers reference point, is when the signal of success is equal to one. As the reader might guess the naive supplier is just a particular case in this model and it arises when $m = 1$.

So m determines the producer's sensitivity to the market states or to the signal of success perceived. In general, all of them behave in the same manner. When $\frac{D_n}{S_n} = 0$, and $\frac{D_n}{S_n} = 1$, there is no change in their behaviors, they expect the demand to be 0 and D_n respectively as we saw in the naive supplier case. The interesting behavior occurs when $0 < \frac{D_n}{S_n} < 1$, and when $\frac{D_n}{S_n} > 1$. In the first subset of outcomes the supplier perceives lower demand in proportion to the quantity supplied at time n. Because of that, he will produce fewer goods than before. His optimism will drive him to produce a little bit more goods compared to what the naive producer would had produced in the same situation. As his m increases the supplier becomes more and more optimistic and he will produce more goods. On the other hand when $\frac{D_n}{S_n} > 1$, the supplier perceives high demand in proportion to the quantity supplied at time n. Therefore, he will produce more goods than before. However, his cautiousness will play an important role. He will produce fewer goods compared to what a naive producer had produced in the same situation. As his m increases he is considered to be more cautious and he will produce less goods. We can write down this model as follow,

$$D_{n+1} = a - bP_{n+1}, \tag{2.13}$$

$$S_{n+1} = \sqrt[m]{(\frac{D_n}{S_n})} \times S_n, \tag{2.14}$$

$$P_{n+1} = \frac{1}{1-M} \cdot (\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2). \tag{2.15}$$

Simplifying this system of equations we obtain the following two dimensional map for the demand and the supply,

$$D_{n+1}(1-M) = a(1-M) - b(\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2). \tag{2.16}$$

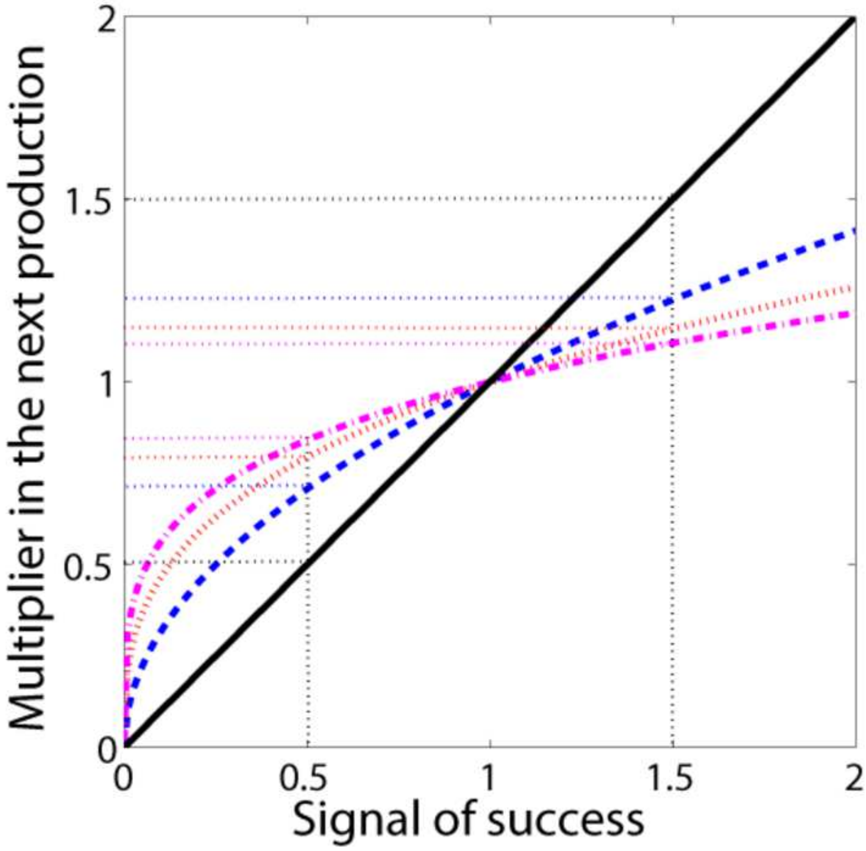

Figure 2.2. Behaviors of suppliers in terms of the signal of success. The relationship between the *nth* root of the signal of success with the multiplier in the next production is shown in the figure above. The solid black curve represents the linear case or the naive supplier, $m = 1$. The blue dash line is the square root $m = 2$ of the signal of success. The red dot line is the cubic root $m = 3$ of the signal of success and the magenta dash-dot line is the $4th$ root of the signal of success. We have plotted the horizontal dot lines, to help the reader see the multiplier of production in each case, when the signal of success is 0.5 and 1.5.

$$S_{n+1} = \sqrt[m]{(\frac{D_n}{S_n})} \times S_n. \tag{2.17}$$

Here the producer needs two seeds to calculate the expected demand, D_0 and, S_0. Notice that this two dimensional map can be reduced into a one dimensional map in terms of supply as shown in Eq. (2.18).

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}(\frac{1}{1-M}(a - b(\frac{F_c}{S_n} + v - vS_n + S_n^2)))} \times S_n. \tag{2.18}$$

2.4 Methodology

We have studied only two variations of the model. Equation (2.19), shows the naive supplier when the parameters are fixed as: $a = 10$, $b = 0.09$, $v = 4$, $Fc = 10$ and $M = 0.5$.

$$D_{n+1}(0.5) = 10 - 0.09(\frac{10}{D_n} + 4 - 4D_n + (D_n)^2). \tag{2.19}$$

Equations (2.20) and (2.21) represent the cautious and optimistic supplier when the parameters are fixed as: $a = 30$, $b = 0.125$, $v = 6$, $Fc = 30$, $M = 0.5$ and $m = 2$.

$$D_{n+1}(0.5) = 15 - 0.125(\frac{30}{S_{n+1}} + 6 - 6S_{n+1} + (S_{n+1})^2), \tag{2.20}$$

$$S_{n+1} = \sqrt[2]{(\frac{D_n}{S_n})} \times S_n. \tag{2.21}$$

We have used the following one dimensional map to compute the Lyapunov exponents spectrum of the cautious and optimistic supplier,

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}(2(30 - b(\frac{30}{S_n} + 6 - 6S_n + S_n^2)))} \times S_n. \tag{2.22}$$

We have studied the dynamics of both models using three tests. First, we have computed the time series of both models to observe the dynamics by applying a recursive algorithm. We have changed the parameters b and M to see how the dynamics of the time series changes. We have chosen to show only the chaotic time series because we want to prove the existence of chaos in the model. Secondly, we have plotted the bifurcation diagrams of the quantity demanded against the parameter b in both cases. At each value of b, we have iterated the functions until they reached the equilibrium points using a recursive algorithm. Then, we have plotted the values of D_{n+1} corresponding to the specific value of b on the same plot. We have done the same with the parameter M in the naive supplier case, to show the dynamics when the margin is changed. Lastly, we have computed the Lyapunov exponents spectrum of both systems.

2.5 Market dynamics

The naive supplier

In order to understand the relationship between the price and the quantity demanded, we have plotted the first 20 periods of trade as shown in Fig. 2.3. We clearly see the price and the quantity demanded behave exactly how we expected. High prices are responded with low demand and low prices are responded with high demand. However, the plots show an irregular behavior in both cases. The economical meaning of this behavior is that the supplier and the customers have not agreed on the quantity and the price during the trade. In other words, their interactions were not translated into market equilibrium. Furthermore, it seems that this market is not efficient. But there is a small window between time steps 6 to 10, in which

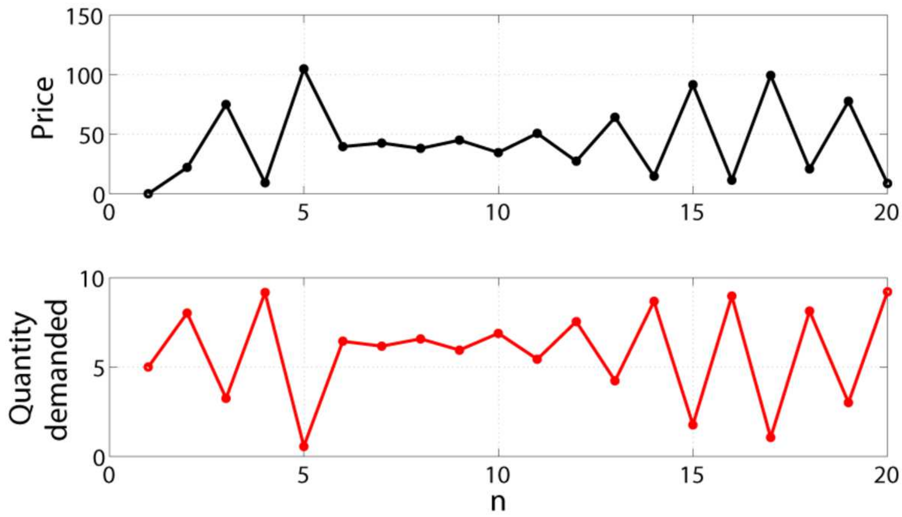

Figure 2.3. The price and demand time series that correspond to the naive supplier during the first 20 periods of trade. The two time series that are shown in the figure above were plotted iterating Eq. (2.18) and (2.10). The black line corresponds to the price, and the red line corresponds to the demanded quantity in the first 20 periods of trade. Despite the fact that the price and the demand are discrete quantities, it is easier to follow their evolution plotting them as continuous curves. But, note that the lines between the dots are meaningless.

the trajectories of the price and the quantity demanded are almost flat or almost in equilibrium. However, after two time steps this behavior changes abruptly into high amplitude fluctuations. We would expect that real world markets of ordinary goods, to behave dynamically and not to fall into the frozen state that standard models predict. We did not obtain this behavior by an accident; we have chosen the parameter values precisely to get this behavior. Next, we will show that more dynamical behaviors are possible computing the bifurcation diagram.

For given values of the parameters b, and M, we can compute the fixed points of the Eq. (2.11). If we allow the parameter b to vary between 0 and 0.0918, we can establish the equilibrium points for D_{n+1}, by plotting the bifurcation diagram of D_{n+1} against b as shown in Fig. 2.4.

The period-doubling route to chaos [60, 61] is obvious looking at Fig. 2.4. We have found period 6 and period 10 cycles when $b = 0.8531$ and $b = 0.0843999995$, respectively. We clearly see the huge range of demand dynamics when we are varying the parameter b. We will explain why this outcome is meaningful in terms of demand theory in the next Section. We obtain a similar bifurcation diagram when we vary M against D_{n+1}. Figure 2.6 shows how the quantity demanded is affected by the gross margin, when it is changed. Notice that in Fig. 2.6, $b = 0.03$. One can check in Fig. 2.4 that at this value of the parameter b, the system should be in equilibrium. Incrementing the gross margin in order to obtain more profits leads to

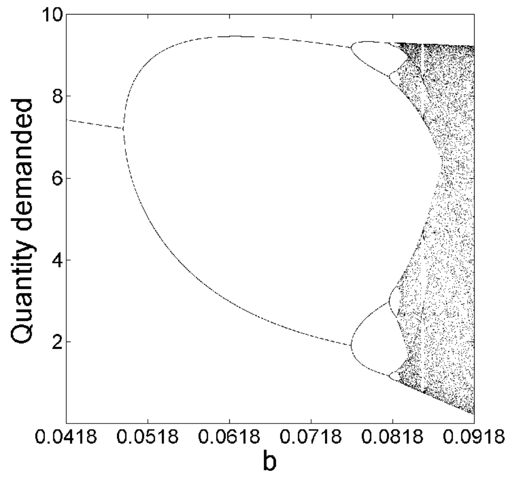

Figure 2.4. The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter b. We have divided the interval $(0.0418, 0.0918)$ of the parameter b into $10,000$ values. Then, we have set each value of the parameter b in Eq. (2.10) and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of the parameter b to obtain this bifurcation diagram.

a destabilization of the whole system. The model suggests that the supplier greed has limits. This is the proof that the supplier has influence on the global dynamics of market. We have also computed the Lyapunov exponent spectrum to prove the existence of chaos as shown in Fig. 2.5.

The cautious and optimistic supplier
We start again with the time series shown in Fig. 2.7.

It is possible to verify how high prices are responded with low demand and vice versa. We can see periods where the demanded and supplied quantities are almost the same. In these periods the system is almost at equilibrium so the price is stable. But after some time the system goes out of equilibrium and periodic-cycles and chaotic behavior arise. We have plotted the bifurcation diagram of D_{n+1} against b to illustrate some more possible behaviors as shown in Fig. 2.8. A period 3 cycle occurs when $b = 0.1308$. This observation implies chaos [35]. We can clearly see that the period doubling route to chaos from Fig. 2.8 as well. Furthermore, we have computed the Lyapunov exponent spectrum to prove the existence of chaos as shown in Fig. 2.9.

2.6 Transient chaos and market collapse

In this Section, we will expand the economical assumptions of the model to emphasize the idea, that a final bifurcation can be a good description of a market collapse. We have chosen the naive supplier as a case study. But the reasoning and

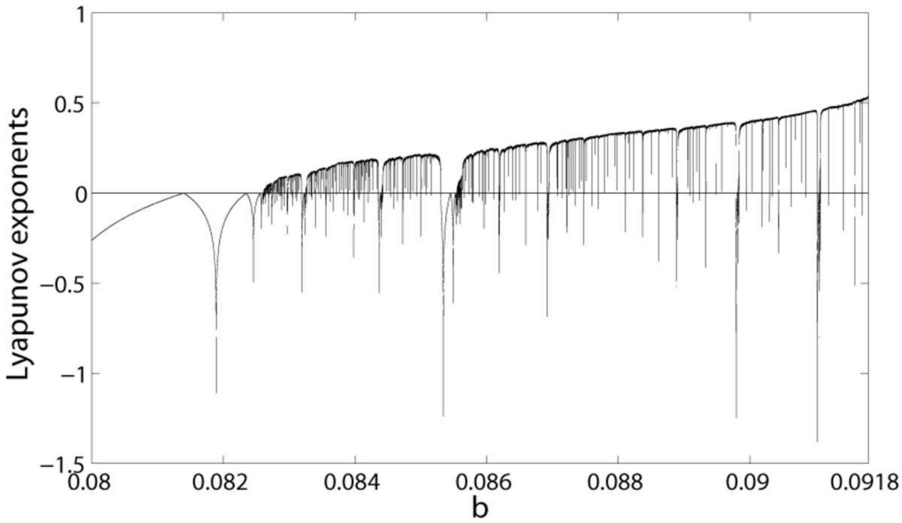

Figure 2.5. The Lyapunov exponent spectrum corresponding to the naive supplier when parameter b is varied. We have taken the interval $(0.08, 0.092)$ of the parameter b and we have computed the Lyapunov exponent of $100,000$ points within this interval. Finally, we have plotted the corresponding exponent against its corresponding value of the parameter b to obtain the spectrum. The exponent is positive in a wide range of parameter b values, what proves the chaotic behavior of the system.

the methodology that we have used to demonstrate this claim, is generic, and can be applied to all types of suppliers.

When the parameters are fixed in Eq. (2.10), and (2.11) as: $D_1 = 1$, $S_1 = 1$, $a = 10$, $b = 0.095$, $v = 2$, $Fc = 20$ and $M = 0.5$, we get the following maps for the demand, the supply and the price :

$$D_{n+1}(0.5) = 10 - 0.095(\frac{20}{D_n} + 2 - 2D_n + (D_n)^2). \tag{2.23}$$

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}(2(10 - 0.095(\frac{20}{S_n} + 2 - 2S_n + S_n^2))) \times S_n}. \tag{2.24}$$

$$P_{n+1} = \frac{\frac{20}{10-0.095(P_n)} + 2 - 2(10 - 0.095(P_n)) + (10 - 0.095(P_n))^2}{1 - 0.5}. \tag{2.25}$$

Analyzing the time series produced by these maps we find a transient chaotic behavior as shown in Fig. 2.10. The trajectories of the quantities demanded, the quantity supplied and the price are completely chaotic until time step 69, where suddenly they explode. By explode we mean the system starts to fluctuate without control giving rise to quantities that are unscaled to the system or even infinitely

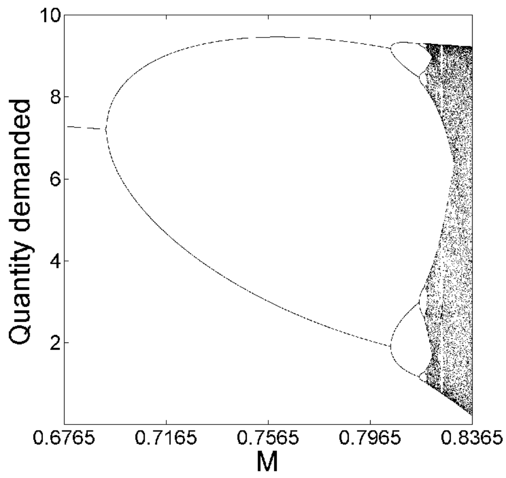

Figure 2.6. The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter M. We have divided the interval $(0.6765, 0.8365)$ of the parameter M into $20,000$ values. Then, we have set each value of parameter M in Eq. (2.10) and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of parameter M to obtain this bifurcation diagram. Notice that when the gross margin is between 0 and 0.6765 the system is in equilibrium. This is a huge range of gross margin values. In contrast, only a small part of the gross margin interval causes the demand to behave chaotically. It is not a surprise that this small part corresponds to high margins.

large. We are not familiar with the complicated concepts of negative infinite price or infinite demand and supply. Therefore, to get a better economical understanding of this situation we need to extend our assumptions about the model.

We will first, focus on the demand side of the system. The meaning of parameter a in Eq. (2.1) is that when the good is freely available (its price is zero) in the market, the maximum amount of goods that can be demanded is the value of the parameter a. This is an accomplished fact, and it is the upper bound of the quantity of goods that can be demanded in this market, assuming the system lies in the positive domain. When we allowed the price to take negative values, the amount of goods demanded was much higher from the value of the parameter a. In this scenario the supplier must pay the consumer to create the demand. We will assume that the supplier does not make strategic decisions thinking on long time horizons. So, when the price is negative he just lose the incentives to supply. Equation (2.26) integrates this new behavior into the model,

$$D_{n+1} = \begin{cases} 0 & if\ (P_{n+1} \times b > a), \\ \mathrm{a} - b \times P_{n+1} & if\ (P_{n+1} \times b \leq a), \end{cases} \tag{2.26}$$

Following the same reasoning as in the demand case, we extend our assumptions on the supply side of the system. The second assumption of the model is that the

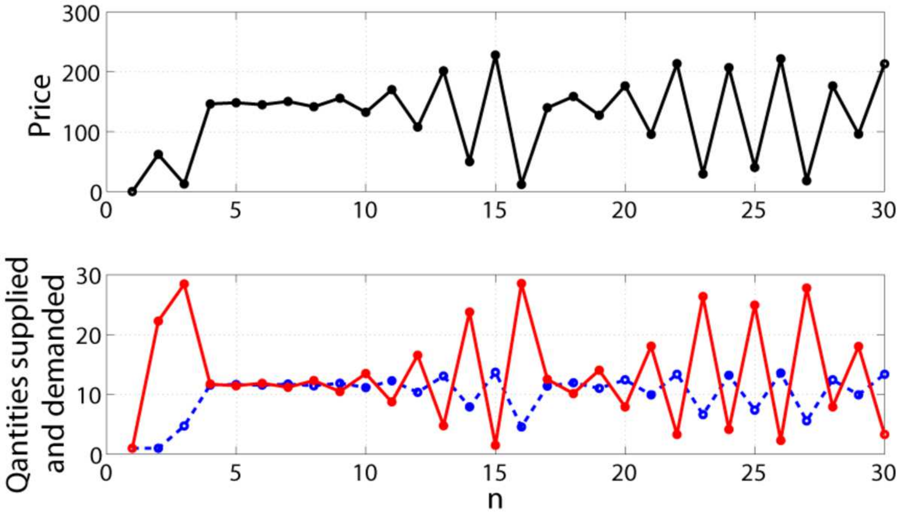

Figure 2.7. Time series of the cautious and optimistic supplier in the first 30 periods of trade. At the bottom we have plotted the demand D as a solid red line against the supply S as a dash blue line. Above in black, we have plotted the price trajectory in this trade scenario. This figure shows the dynamic behavior of the quantities supplied and demanded, and the price. The price is moving exactly as we would expect. There are periods where the price does not change much, so we can say the market is almost in equilibrium. And there are periods where the price changes dramatically, what corresponds to the nonequilibrium state of the market.

supplier always tries to sell exactly the amount of goods he produced or bought. If he expects zero or negative demand we can assume the supplier will not produce anything for the next period of time. He will probably get out of the market in this situation. The supplier computes the expected demand before going into production, so if he sees that the expected demand is zero or negative he stops immediately the process. We can describe mathematically this behavior using Eq. (2.27),

$$S_{n+1} = \begin{cases} \frac{1}{1-M} \cdot (\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2) \text{ if } D_{n+1} > 0, \\ \textit{stop} \text{ if } D_{n+1}, \leq 0. \end{cases} \tag{2.27}$$

When the trajectories arrive to the final bifurcation the market stops to exist immediately. The reader can see in Fig. 2.10, how after the final bifurcation the price stays at some high level where the quantities supplied and demanded go to zero. Note that if the demand crosses some critical value (small value), the system enters into a loop of destruction, because of the growing cost of production of diminishing quantities. We would expect similar dynamics in a situation of market collapse.

In real economies we find two interesting properties that can be also observed in this model. The first one is the prediction problem, in which the collapse is impossible to forecast beforehand. Secondly, the global complexity of the market emerges from simple nonlinear interactions between the economical agents.

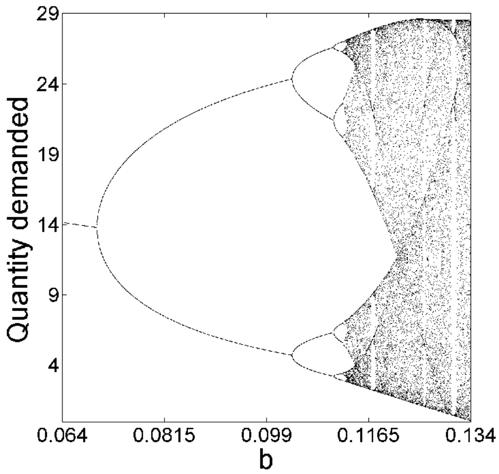

Figure 2.8. The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter b. We have divided the interval $(0.064, 0.134)$ of the parameter b into $10,000$ values. Then, we have set each value of the parameter b in Eq. (2.16) and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of the parameter b to obtain this bifurcation diagram.

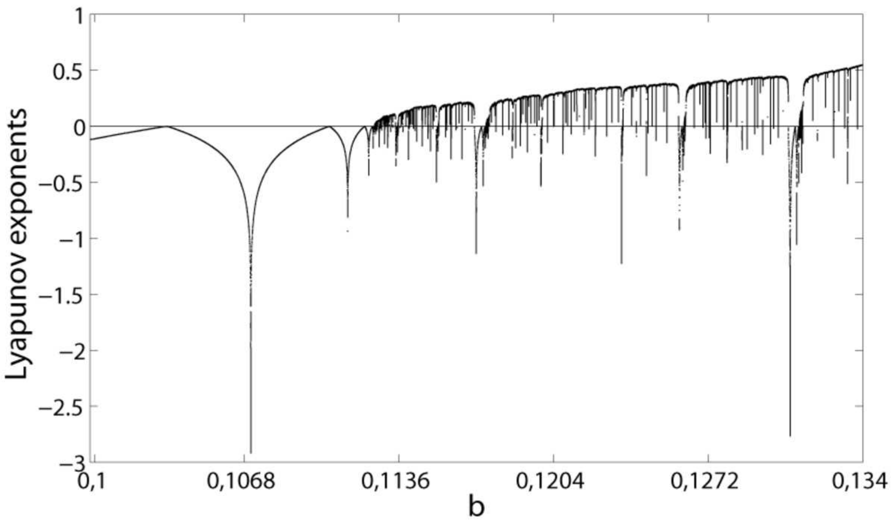

Figure 2.9. The Lyapunov exponent's spectrum corresponding to the cautious and optimistic supplier when parameter b is varied. We have taken the interval $(0.1, 0.134)$ of the parameter b and we have computed the Lyapunov exponent of $80,000$ points within this interval. Finally, we have plotted the corresponding exponent against its corresponding value of parameter b to obtain the spectrum. The exponent is positive in a wide range of parameter b values, what proves the chaotic behavior of the system.

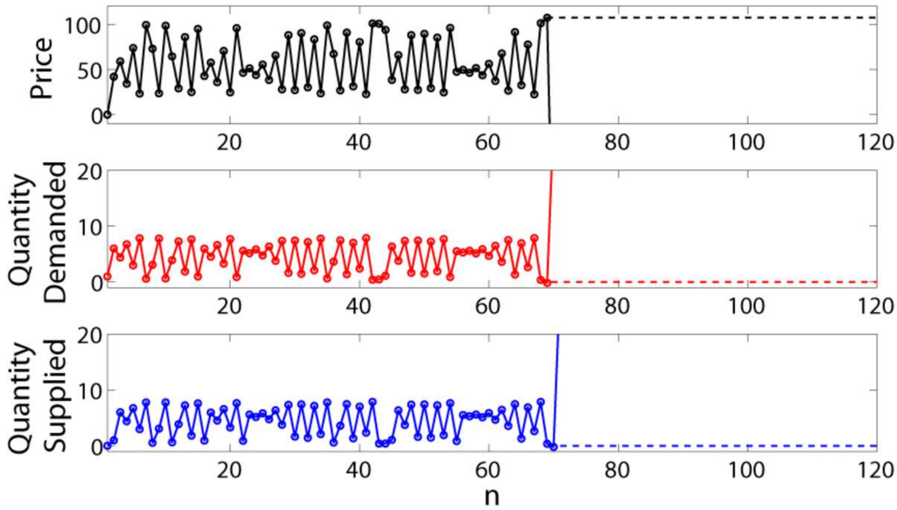

Figure 2.10. Time series of the quantities demanded and supplied before and after bounding the system. The solid line represents the time series of the price, the demand and the supply, simply by iterating the maps fixing the parameters as: $D_1 = 1$, $a = 10$, $b = 0.095$, $v = 2$, $Fc = 20$ and $M = 0.5$. The time series behaves chaotically until time step 69 where a very big fluctuation occurs. The price becomes negative so the quantities demanded and supplied increase dramatically. The dash line represents the same system as before but now bounded. The time series can not be negative so that, when some critical value is crossed the system simply goes to zero, as in the case of the quantity demanded and supplied shown in the figure above.

2.7 Parametric analysis

We have modeled the demand as a monotonic function. Nevertheless, the slope of the demand curve, parameter b, has a huge effect on the dynamics of the system as we saw in the previous sections. To capture this idea we can compute the *price elasticity of demand* (PED), which measures the quantity demanded sensitivity to the price and it is given by the following ratio:

$$PED = \frac{\%\text{change in Quantity demanded}}{\%\text{change in Price}}. \tag{2.28}$$

In general, goods which are elastic tend to have many substitutes, they must be bought frequently and they assume to be traded in a very competitive market. In this model we have assumed all above. We have done this by modeling the market as an ordinary good market that obeys the demand law. When we vary the parameter b, we change the price elasticity of demand. For example, when $b = 0$, we encounter a perfectly elastic demand curve. One can imagine the demand curve as an horizontal line. At this certain price the demand is infinite, so any amount of goods is quickly consumed. In Fig. 2.11 we clearly see how the quantity supplied in blue is rapidly sticking to the quantity demanded in red until all the demand is fulfilled. Due to

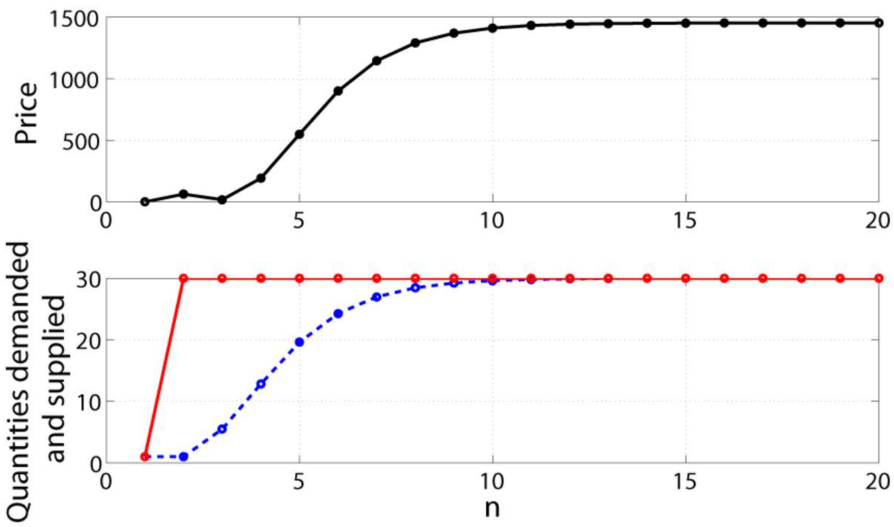

Figure 2.11. Dynamics of supply when there is a perfectly elastic demand curve. Time series of the first 20 periods of trade in the cautious and optimistic producer case when $b = 0$. At the bottom we plotted the demand D in red against the supply S in blue. Above we plotted the price trajectory of this trade scenario.

the excess demand the price is going up until it reaches the market equilibrium price. This process is not instantaneous as can be checked. Even though we have assumed a perfectly elastic demand, the supplier does not know it. It takes him about 13 periods of trade to supply all the goods demanded by the market. This is a good example of the adjustment dynamics that underlies the market equilibrium assumption.

But the really remarkable result is that a very small change in the PED can change completely the system dynamics. Figure 2.8 describes how the global dynamics of the system changes as we increase the value of the parameter b inside a very small subset. When $0 < b < 0.134$, we observe equilibrium points, cycles and chaotic trajectories, but when $b > 0.134$, the system explodes. We have showed in the previous Section that the economical meaning of this exploding dynamics is a market collapse.

This behavior is not special only for b, when the value of M and a are varied, we encounter the same dynamics, but we assume that the gross margin value is controlled or partially controlled by the supplier. Therefore, theoretically the supplier can avoid erratic trajectories or crash scenarios manipulating this variable. We have focused on the price elasticity of demand because it cannot be influenced by the supplier but it is directly related to the price. Exactly like in a real world, the small supplier tries to adjust its production to the demand, and not the demand to the production. Because trying to influence the demand is highly expensive and only big companies with more resources can afford it.

2.8 Conclusions

We have introduced the supply based on demand model studying two types of suppliers, the naive supplier and the cautious and optimistic supplier. In both cases we have found that the model is capable of reproducing a large variety of dynamics such as equilibrium, limit cycles, chaos, and even catastrophic dynamics under simple and reasonable economic assumptions. We have emphasized the idea that the final bifurcation can be a good description of a market collapse by adding some new assumptions to the model. We have shown the important role that the price elasticity of demand plays on the global dynamics of the market. One important result is that very small changes in the price elasticity of demand leads to very different global dynamics assuming a monotonic demand function. We have also demonstrated the huge influence of the gross margin, M, on the market dynamics.

Chapter 3
Preventing the crash with partial control

> *"All stable processes, we shall predict. All unstable processes, we shall control."*
>
> -John Von Neumann

Market crashes are some of the most catastrophic events in modern societies. The wounds left after the last financial crisis of 2008 are not healed yet, but they serve us as a reminder that our economic system is very vulnerable to this type of events. An extensive literature exists around the economic crashes phenomenon and it is focused on three main lines of research. The first one analyzes the different causes that lead to this type of crisis in the first place. The second line of research focuses on predicting this kind of events by using different analytical techniques. And the third line of research studies different ways in which the economic system can be controlled to prevent the crash or to minimize its devastating effects. In this chapter we follow the third line of research, and we apply successfully for the first time in economic context the partial control method. This control technique has been used to control many systems in physics, ecology and biology. But in this chapter, we show that it is possible to prevent a crash controlling the market, without any central control, applying much smaller control than the noise that exists in the market.

3.1 Introduction

Economic dynamics constitutes an important research field in economics. Many models have been developed to explain the motion of economic variables such as the price, the demand or the GDP, giving rise to different dynamical behaviors, like periodic orbits, strange attractors and equilibrium states. But, sometimes, extreme events lead to a market collapse. Economists agree that market collapses are charac-

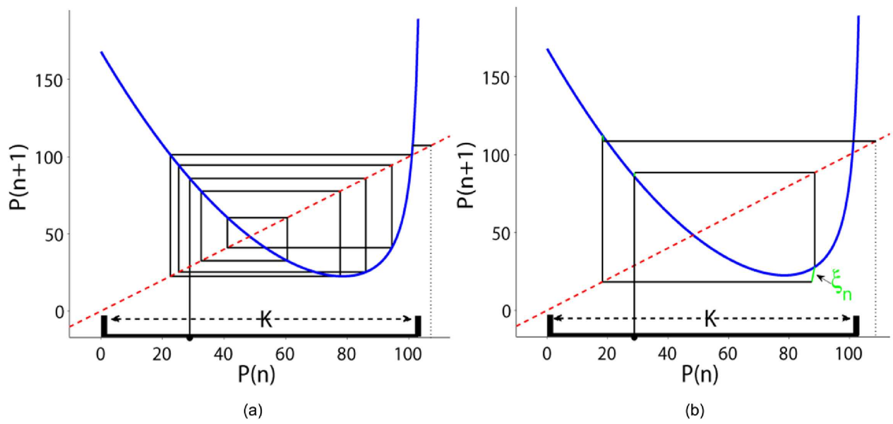

Figure 3.1. Cobweb plots of the price map. The blue line represents the price map, the diagonal red dashed line represents $P_{n+1} = P_n$. The initial condition in both panels is $P(1) = 28.8$. When no disturbances nor control are present in the system, after a few generations the trajectory escapes from the chaotic saddle (region K) towards minus infinity (black dotted line) as shown in panel (a). When a disturbance, $\xi_0 = 10.0$ is present in the system the trajectory escapes from the chaotic saddle even faster as shown in panel (b). The green lines represent the amount of disturbance introduced into the system at each time step. Notice that each line has a different length because it is defined by a uniform distribution function bounded by $\xi_0 = 10.0$. The black thick line over the horizontal represent the region K which is the region where we want to sustain the dynamics.

terized by an abrupt fluctuation or a chain of fluctuations that decreases the value of some economical variable dramatically.

In this chapter, we consider a particular dynamical behavior called *transient chaos*. This phenomenon can be found in many systems such as a thermal pulse combustor [62] a periodically driven CO2 laser [63], a voltage collapse [64] or a three-species food chain ecological model [65]. In economics, transient chaos can be found in many systems as well, such as speculative markets models [66, 67, 68], a business cycle model [69] and a duopoly model [70]. The topological structure behind this behavior is the presence of a chaotic saddle in the phase space. This topological object arises when a chaotic attractor collides with its own basin boundary producing a transient chaotic behavior of trajectories before eventually escaping towards an external attractor [71] as shown in panel (a) in Fig. 3.1.

A collapsing market behaves similarly to a transient chaotic system, where the fluctuations of the price, the demand or the supply are erratic but bounded, until they reach to some critical value after which the whole system collapses as shown in Fig. 3.2. We have chosen the supply based on demand (SBOD) model to study transient chaos in the economy [72]. This model is based on the classic cobweb model [52, 53], with the difference that the firm tries to adjust the production in accordance with the expected demand, instead of the expected price. In some way this model is an iterative deterministic version of the Newsvendor problem [73]. The main differences between these two models are that in the SBOD model, we assume a deterministic demand function that depends on the selling price instead of some random demand function. Additionally, we assume that the process of stocking is an iterative process. The interest of this model relies on the simple explanation of how small firms prepare their inventory for the coming sales season computing the expected demand using a simple model and some past sales data. This model produces the following dynamics: equilibria points, periodic orbits and chaotic behavior, which for some parameter values becomes transient. In this situation, the trajectories of the price, the demand or the supply are chaotic some time until they eventually collapse. In the context of our model, we mean by market collapse a market state characterized by high prices in which the firm loses the incentives to supply due to the very low or even zero expected demand. When the supply and demand vanish the trade becomes impossible and the market collapses. This behavior is shown in Fig. 3.2.

When the system falls in the market collapse state, the question that naturally arises is the possibility of avoiding it, maintaining the system in the transient regime. Three problems arise when trying to control any economical system. The first one is the prediction problem. How do we know beforehand that the market is close to a collapse? A lot of research has been done to answer this question. A few interesting works focused on the stock markets can be found in [26, 74, 75, 76, 77, 78]. The second one is where to apply the control. In many models the control is applied on some parameter that is not trackable or can only be influenced theoretically. The third problem is that all real economies are affected by certain external disturbances, producing large deviations in a nonlinear deterministic system as shown in panel (b) in Fig. 3.1. In fact, many control methods that are effective without disturbances, can fail when the disturbances are present [79]. We have chosen this specific model because it has some parameters that can be easily controlled by the firm. For example, the selling price of the product can be easily adjusted by the firm, changing the firm's gross margin at each time step.

In recent years, a novel control method called partial control has appeared in the literature [80, 81]. This control method is applied in situations where transient chaos is present and the system is subjected to external disturbances as shown in Fig. 3.1. The main results of this chapter are that the firm can successfully control the trajectories of the price by only changing the gross margin at each time step preventing a market collapse. It can also rationalize the quantity supplied with the same purpose. Moreover, we show that firms with market power can influence the demand in the retailer or wholesaler markets, generating market stability in the long run. Furthermore, we prove that the amount of control needed in those cases is even smaller than the disturbance.

The structure of the chapter is as follows. Section 3.2 is devoted to the description of the supply based on demand model. The main ideas of the partial control method

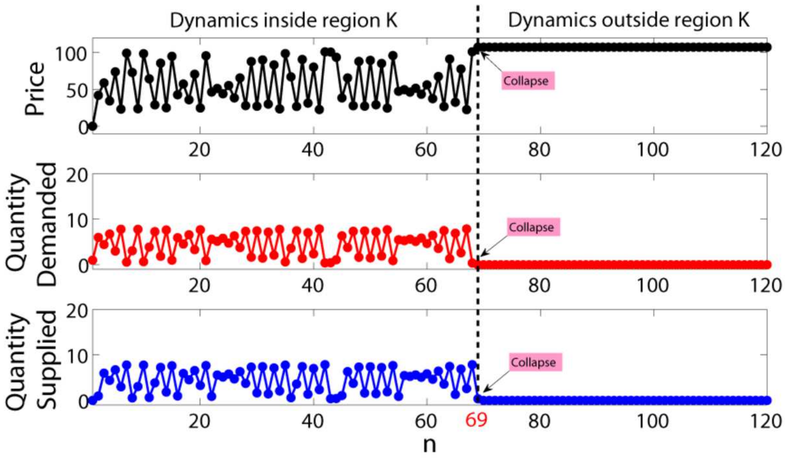

Figure 3.2. Time series of the quantity demanded, the quantity supplied and the price. Two versions of the SBOD model are represented in each panel on this plot, the unbounded version as dashed line versus the bounded version as solid line [72]. Although these two versions are qualitatively identical, so we can not find any differences in the first 69 iterations, their visual representation after time step 69, where the market collapses, is very different. In the unbounded version (dashed line) a big fluctuation occurs after time step 69, while in the bounded version (solid line) the line stays unchanged. In the context of this chapter, we mean by market collapse a situation where after some critical price value is crossed, the quantity demanded goes to zero. Owing to the absence of demand the firm losses the incentives to produce and the market disappear. We are use to associate a market crash with the bounded version kind of plots, in the following sections we will use it for our visualizations.

are described in Section 3.3. In Section 3.4 and 3.5 the safe sets are computed for the price and the quantity demanded in the naive supplier case, in order to produce controlled trajectories. In Section 3.6, we have computed the safe sets of the quantity supplied in the cautious and optimistic supplier case, generating controlled trajectories. Finally, some conclusions are drawn in Section 3.7.

3.2 The SBOD model

We use the supply based on demand model proposed by Levi et al [72]. This model describes the price-quantity dynamics in a market where the consumer obeys the demand law and the firm prices its products by only adding its gross margin to a quadratic average total cost function. The main assumptions done in this model are that the firm is a price maker, implying that it is the only one who sets and adjusts the price in light of circumstances, while its main goals are to sell all the produced products and to satisfy the overall demand. The general structure of the model is as follows,

$$D_{n+1} = a - bP_{n+1}, \tag{3.1}$$
$$S_{n+1} = D^{Exp}_{n+1}, \tag{3.2}$$
$$P_{n+1} = \frac{ATC}{1-M}. \tag{3.3}$$

The quantities demanded and supplied, D_{n+1}, and S_{n+1}, and the price, P_{n+1} are assumed to be discrete functions of time. The parameters a and b are positive constants $a, b \geq 0$. The firm expected demand functional is D^{Exp}_{n+1}, and M is the gross margin added by the firm to obtain profits, where $0 \leq M < 1$. The average total cost functional ATC of the firm will adopt a U-shape, when diminishing returns are present in the production process and the firm has variable costs. Applying this idea, when the firm increases the amount of production the average total cost of every unit of production decreases until it reaches some critical point from which every additional produced product will increase the unit average total cost. In the decreasing side of the curve, the firm enjoys of scale economies, that is, decreasing returns to scale. After crossing this point every additional product produced increments the average total cost of the firm which implies a diminishing returns to scale [82]. The U-shape of the ATC function as shown in Fig. 3.3 captures this idea. In our case, the quantity of production is the same as the quantity of supply, S_{n+1}, or the expected demand estimated by the firm, as shown in Eq. (3.2) and (3.4),

$$ATC = \frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2. \tag{3.4}$$

We assume that v and F_c, are positive constants. In order to obtain profits, the supplier adds over the average total cost of the product some quantity using the gross margin operator shown in Eq. (3.3). When M increases, the ATC function moves upwards, what leads to higher selling prices and when it decreases the ATC moves downwards what leads to a cheaper products as shown in Fig. 3.3. We assume that the selling price is in fact the market price.

The firm makes the decision of what quantity to supply, S_{n+1}, before it knows the reaction of the market to the price it fixes. The firm makes this decision based on the quantity it expects the market will demand, D^{Exp}_{n+1}, in the future. The firm does not know anything about the demand function. The only available information it has, is the quantity demanded at the price in which its products were sold in the last sales season. It is important to notice that the quantity demanded from the firm perspective is the sum of the total units sold and the total units out of stock (stock rupture). The firm quantifies its success after each sales season using a very simple model - it divides the quantity demanded by the quantity supplied at time n as shown in Eq. (3.5). We called it the signal of success S,

$$S = \frac{D_n}{S_n}. \tag{3.5}$$

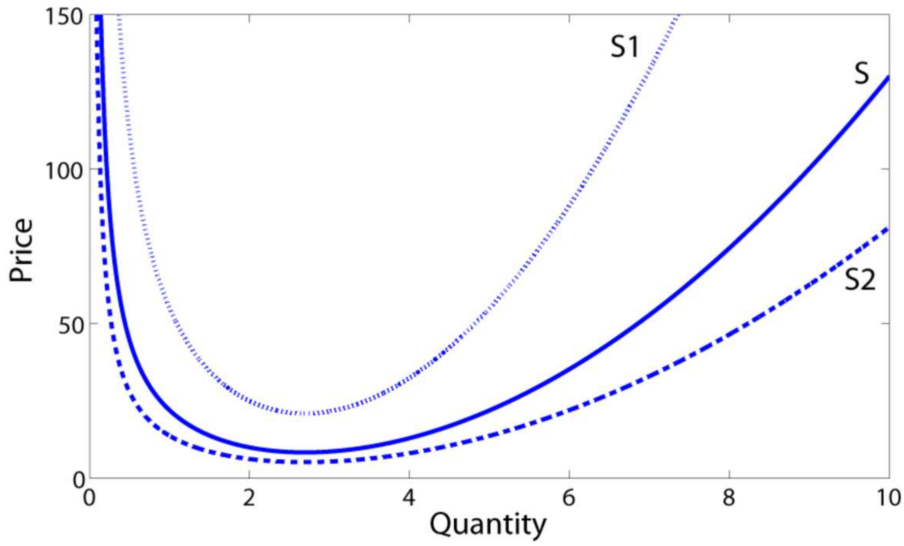

Figure 3.3. The price-quantity function. We have used the following function $P = \frac{1}{1-M} \cdot (F_c Q + v - vQ + Q^2)$, to relate the price of the product with the quantity supplied, where P is the selling price of the product (cost + profits) and Q is the quantity of production. The average fix cost function is $\frac{F_c}{S_{n+1}}$ where F_c is a positive constant and the average variable cost function is $v - vS_{n+1} + (S_{n+1})^2$, where v is positive constant. The parameters are fixed as: $F_c = 10$ and $v = 4$. The supply curves S as solid line, $S1$ as dot line and $S2$ as dash-dot line, correspond to the gross margin $M = 0.5$, $M = 0.8$, $M = 0.2$ respectively. When the firm increases the gross margin M, the price increases and when the firm lows the gross margin M the price decreases.

According to the signal of success, the firm makes the decision of how many products to produce and supply in the next sales season. We assume an ordinary goods market in which, when the price increases, the consumption decreases and vice versa. For simplicity, we assume a linear demand curve with negative slope as shown in Eq. (3.1).

The model describes just one firm, and it does not take into account its financial constraints. Furthermore, the firm does not try to maximize its profits nor accumulate stock. The model works as follows. In the first step the firm supplies a certain amount of products to the market to get some feeling about the demand (seed). Then, it observes the amount of products that were demanded at this specific selling price. According to this quantity the firm decides how many products to produce for the next sales season using a simple model that quantifies the firm success. We called it the signal of success, and is a simple ratio between the quantities demanded and the quantity supplied at time n. The firm uses this signal to estimate the expected demand in the next period. The second step is the pricing process. The firm uses its ATC function to compute the products average total cost. After obtaining the cost per unit, it adds profits over the cost using the gross margin operator. Finally, it introduces the products with their new price into the market, it waits some time until it sees how many products have been sold and then it repeats all the process again at every time step. This model aims to explore the global dynamics of the market as a result of this simple behavior of the firm.

In [72] the authors focused only on two supplier types, the naive supplier and the cautious and optimistic supplier. In this chapter, we will show that the partial control method can be successfully used in both cases.

3.3 The partial control method

The partial control method has been successfully applied to several paradigmatic dynamical systems, such as the Hénon map, the tent map [83], the time-2π map associated to the Duffing oscillator [84, 85, 86] and the 3D Lorenz map [87]. In particular, when we consider the dynamics after a boundary crisis, the system possesses a transient chaotic behavior in a bounded region in phase space, previous to a situation in which the trajectory escapes towards an attractor outside this region. When the dynamics is affected by noise, somehow it might help the trajectory to escape from the region earlier as shown in Fig. 3.1. The goal of the partial control is to apply a control in order to avoid the escape of the trajectory from this region K, and what is surprising is that the amount of control we need is smaller than the external disturbance acting on the dynamical system. To implement this method, we need a map and to define a region K in phase space, where we want to sustain the dynamics. The complete dynamics in presence of an external disturbance ξ_n and after the application of a control u_n is described by the iterative equation $k_{n+1} = f(k_n) + \xi_n + u_n$. The only assumption we consider on the disturbances and control is to be bounded, that is, $|\xi_n| \leq \xi_0$ and $|u_n| \leq u_0$, and when this happens we say that we have admissible disturbances and controls. A point $k \in \mathrm{K}$ is considered safe, if the next iteration of this point $f(k)$ under the action of the map and affected by the external disturbance can be put again on K once a control $|u_n| \leq u_0 < \xi_0$ is applied. We can say that under the previous considerations, a safe point is partially controlled and consequently remains in K with an applied control smaller than the disturbance. The set of all safe points in K is called the safe set. There is an algorithm called *Sculpting Algorithm* [85, 86], that computes automatically (if it exists) the safe set given a map, a region K in phase space and admissible disturbances and controls. Our goal here is to compute the safe set for the supply based on demand model for the naive case described in the previous chapter. The Sculpting Algorithm works in such a way that it rejects, in the first iteration, the points k_n for which $k_{n+1} = f(k_n) + \xi_n$ need a control $\xi_n < u_n$ to get back to the region K. The points that survive are a subset of K, and the process is repeated until it finally converges. As a result, we obtain the safe set containing all safe points, which is formed by those points that are controlled with admissible disturbances and controls.

3.4 Controlling the price

Here, we will demonstrate how the naive supplier can prevent a market collapse by only applying the partial control strategy on the selling price. From the naive supplier model shown in Eqs. (3.6-3.8), we derive the map for the price in Eq. (3.9) to which we will apply the partial control method,

$$D_{n+1} = a - bP_{n+1}, \tag{3.6}$$

$$S_{n+1} = D_n, \tag{3.7}$$

$$P_{n+1} = \frac{1}{1-M} \cdot (\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2). \tag{3.8}$$

Our goal is to find the safe sets for the following map of the price,

$$P_{n+1} = \frac{1}{1-M} \cdot \Big(\frac{F_c}{a - bP_n} + v - v(a - bP_n) + (a - bP_n)^2\Big). \tag{3.9}$$

We have chosen the parameter values as follow, $M = 0.5$, $F_c = 20$, $v = 2$, $a = 10$ and $b = 0.095$. These parameter values correspond to the region after the boundary crisis, since we are interested in the transient chaotic regime as we saw in the previous section. In order to apply the Sculpting Algorithm to find the safe sets, we need to define a region K in the phase space where the map is acting and where we want the dynamics to stay in. Then, we can compute an admissible choice of disturbances and controls.

We know that the iterates of any initial point for which $P_n > P^*$, follow chaotic dynamics until they finally asymptotes to infinity when they cross the critical value $P_n < P^*$, which actually implies uncontrolled growing price fluctuations. We have defined the initial region K in the phase space where we want to maintain the dynamics of the system as follows. There is a critical price value $P^* \simeq 48.838$, in which if the supplier prices the product above it, the market will collapse in some close future. The upper bound of the region K is subject to the production function of the supplier, and the lower bound of the region K is assumed to be zero. The critical price value $P^* \simeq 48.838$ must be inside the region K to ensure the success of the control strategy. We have chosen for our simulations the initial region K to be the interval $P_n \in [0, 103]$ see Fig. 3.4. Note, that region K contains the chaotic saddle, which is the responsible for the existence of the chaotic transient.

Furthermore, we have chosen a uniform noise distribution bounded by ξ_0, where a disturbance $|\xi_n| \leq \xi_0$ is introduced each time step. The reader can think of the disturbance as any unpredictable positive or negative change in the price, that was not taken into account in the pricing process. Oil prices may be a good example for that. Consider a situation where the price of oil suddenly goes up, incrementing the transportation costs. This random fluctuation will influence immediately on the selling price. When the firm did not have time to change its margin or the variable costs considering this unpredictable fluctuation, it can influence the price

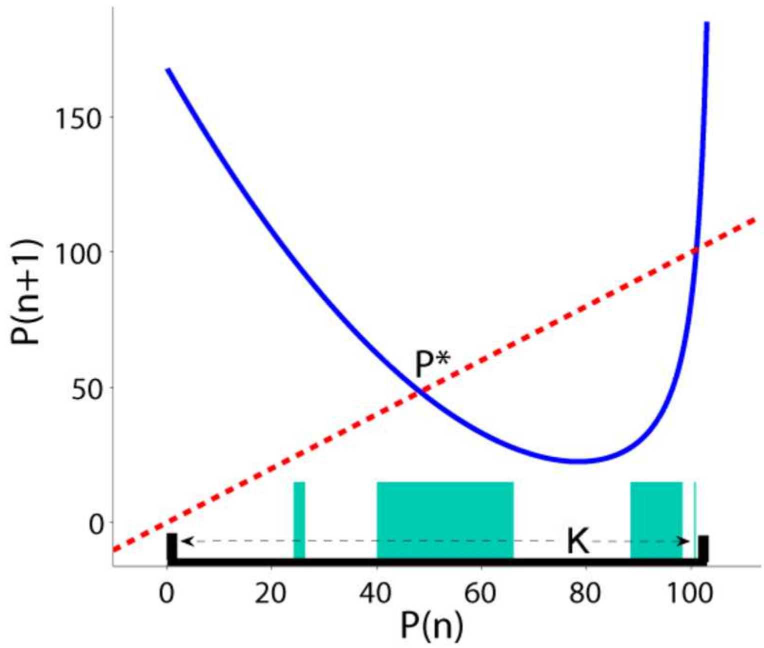

Figure 3.4. The phase space of the price map. This figure shows the phase space of the price map (blue line) $P_{n+1} = f(P_n)$ and the region K (black thick line over the horizontal) where we want to sustain the dynamics. Using the Sculpting algorithm we have computed the safe sets that correspond to an admissible choice of disturbances and controls. Then, we have plotted them as turquoise rectangles over the region K.

applying some control. The control term in contrast is not random at all. The firm applies it with the only purpose of controlling the price trajectory avoiding the market collapse. We want to remind that the firm controls the price without the intend of maximizing profits, he uses this control method only with the objective of maintaining the "business alive". The firm makes discounts when the price is high or inflate the price when the price is low, in order to control the long term trajectory of the price. Those ups and downs in the price affect only the gross margin of the supplier. The new margin is easy to compute including the control term.

Now, we can use the Sculpting Algorithm [85] in order to find the safe sets. The computation of the safe set depends on the chosen values of ξ_0 and u_0 and our observations indicate that for a given ξ_0, we may obtain different safe sets which correspond to different values of u_0. The lower the u_0 bound the smaller the final safe set is. Nevertheless, there is a critical value of u_0, below which no safe set exists. We have chosen for our numerical simulations $\xi_0 = 10$ and $u_0 = 6.82$, where u_0 is very close to the minimum value for which safe sets exists. When the trajectory crosses the critical value P^*, we evaluate the value of $f(P_n) + \xi_n$. If the point is inside a safe set we do not apply the control, and if it is outside, we relocate it inside the nearest safe point, resulting the new safe point $P_{n+1} = f(P_n) + \xi_n + u_n$. The final result of this computation gives rise the safe sets as shown in Fig. 3.4.

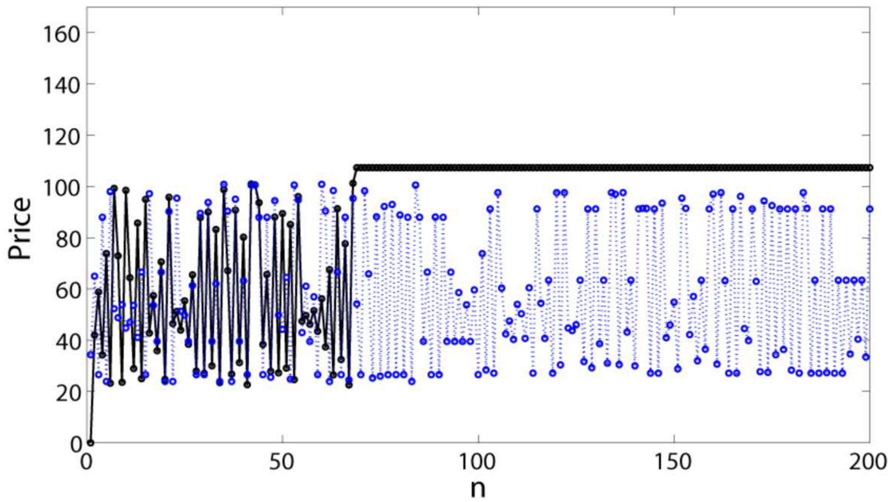

Figure 3.5. Controlled time series of price. Black line: time series of the price without control exhibiting an escape towards some high price level in which no trade can be done. Blue dot line: controlled time series of the price where the market collapse is avoided. This time series corresponds to the first 200 iterations of the system.

The time series displayed in Fig. 3.5 shows clearly how the firm prevents the price in which the trade becomes impossible using the partial control method. The reader can see the controlled trajectory in blue versus the uncontrolled trajectory in black. Furthermore, the amount of control needed at each time step to maintain the dynamics of price in the transient regime is much smaller than the disturbances, as shown in Fig. 3.6.

3.5 Controlling the quantity demanded

The demand is much more difficult to control than the price, because it is a variable that depends on other preferences and actions and it is an unaccessible variable to the agent who tries to control it. Moreover, driving the demand is a very expensive task, and it might be done only by the most powerful agents in the economy, such as, large market share companies or the government. We can apply the partial control method in two different conceptual frameworks of our model, the retail market and the wholesale market. In the retail market the firm can influence the demand directly, using a massive advertising and promotional campaigns or even buy its own goods at the market price when there is an excess supply. When a powerful firm is sitting on the demand side of the model representing the entire demand for a much smaller firm, we are modeling a wholesale market situation. Intuitively, these two firms depend completely on each other. The small firm has only one client - the powerful firm. In the same time the powerful firm has only one supplier - the small firm. The powerful firm can force the small firm to match its production with its own needs, but it must be very careful in not stressing the small firm too much

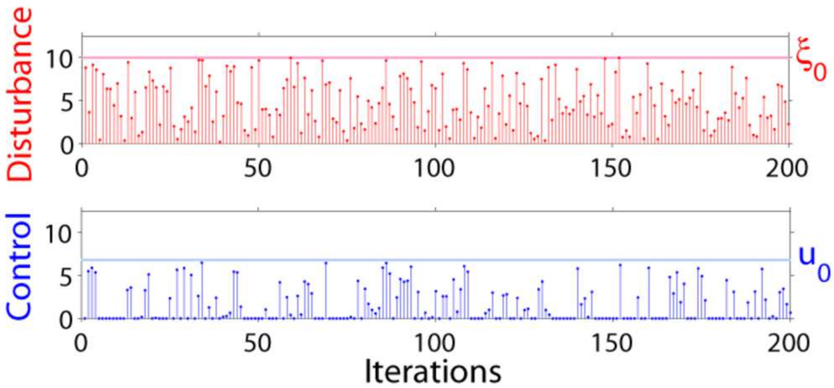

Figure 3.6. The disturbance and control applied in absolute values at each time step. Note that the control applied in order to sustain the trajectory in the transient regime is always smaller than the disturbance. The reader can check that every time step the amount of control (blue bar) is much smaller than the amount of the disturbance (red bar), where the average control disturbance ratio is 0.66.

due to the catastrophic result of that action. In this context, the collapse of both firms is a sufficient motive of controlling the demand in the system. Technically speaking, it is simple to apply the partial control method to solve these problems in the context of our model and it can be done in a very efficient way. As in the previous section, we will begin with finding the safe sets. We derived the map for the demand in Eq. (3.10) from the naive supplier model presented in Eqs. (3.6-3.8). We have followed the same strategy as before to compute the region K. The firm estimates the potential amount of goods that can be consumed in the market D_{max}. As we can see in Fig. 3.7, almost every initial demand D_0 sooner or later is repelled to minus infinity. Thus, no matter what quantity is chosen by the firm for D_{max}, the system will lay on a transient chaos regime. We have chosen for the numerical simulations the initial region K to be the interval $D_n \in [0.4, 8]$ as shown in panel (c) in Fig. 3.7,

$$D_{n+1} = a - b\Big(\frac{1}{1-M} \cdot \big(\frac{F_c}{D_n} + v - vD_n + (D_n)^2\big)\Big). \tag{3.10}$$

We have introduced a noise term to the system exactly as in the previous section. This disturbance represents an unexpected demand. For example, an unpredictable new trend or an unpredictable seasonal effect. If the firm sees that a positive control is needed, it might intervene directly in the market, buying the indispensable amount of products that ensure the demand to be met. Massive promotional and advertising campaigns can be used to achieve the same goal in an indirect manner. It is clear that, those two possibilities are very expensive, hence, just powerful firms can afford such expensive interventions in the economy. As in the previous section, there is a critical value of u_0, below which no safe set exists. We have chosen for our numerical

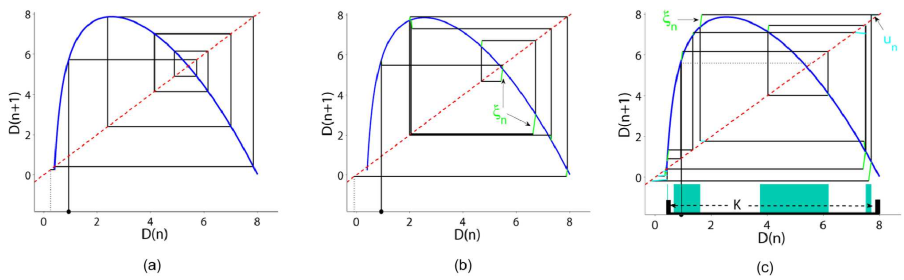

Figure 3.7. Cobweb plots of the demand uncontrolled and controlled orbit. The blue line represents the demand map, the diagonal red dash line represents $D_{n+1} = D_n$. The initial condition in all panels is $D(1) = 0.932$. When no disturbances nor control are present in the system, after a few time steps the trajectory escapes from the chaotic saddle towards minus infinity (black dotted line) as shown in panel (a). When a disturbance, $\xi_0 = 1.0$ is present in the system the trajectory escapes from the chaotic saddle even faster as shown in panel (b). The green lines represent the amount of disturbance introduced into the system at each time step. Notice that each line has a different length because it is defined by a uniform distribution function bounded by $\xi_0 = 1.0$. Panel (c) shows the region K (black thick line over the horizontal) which is the region where we want to sustain the dynamics. Using the Sculpting Algorithm we have found the safe sets and we have plotted them as turquoise rectangles over the region K. A control term bounded by $u_0 = 0.66$ is applied each time step (cyan lines) preventing the collapse.

simulation $\xi_0 = 1.0$ and $u_0 = 0.66$, where u_0 is very close to the minimum value for which safe sets exists. When the trajectory is out of the safe set, we evaluate the value of $f(D_n) + \xi_n$. If the point is inside a safe set, we do not apply the control, and if it is outside, we relocate it inside the nearest safe point, resulting the new safe point $D_{n+1} = f(D_n) + \xi_n + u_n$. The final result of the application of the Sculpting Algorithm is shown in Fig. 3.7.

Although in practice there are plenty of difficulties to estimate the demand in the market, the reader can check in the time series in Fig. 3.8 how a powerful firm can actually prevent the demand of crossing the critical point that triggers the disintegration of demand and subsequently the collapse by using the partial control method. Furthermore, the amount of control needed at each time step to maintain the dynamics of demand in the transient regime is much smaller than the disturbances as shown in Fig. 3.9.

As we mention in the previous sections, the quantity supplied in the time step $n + 1$ is simply the quantity demanded at time step n. Thinking about this, the supplier may apply the partial control method directly on the quantity supplied.

3.6 Controlling the quantity supplied

When the future demand is uncertain, controlling the quantity of supply is a very complex task. In our model we assume a deterministic process where after computing

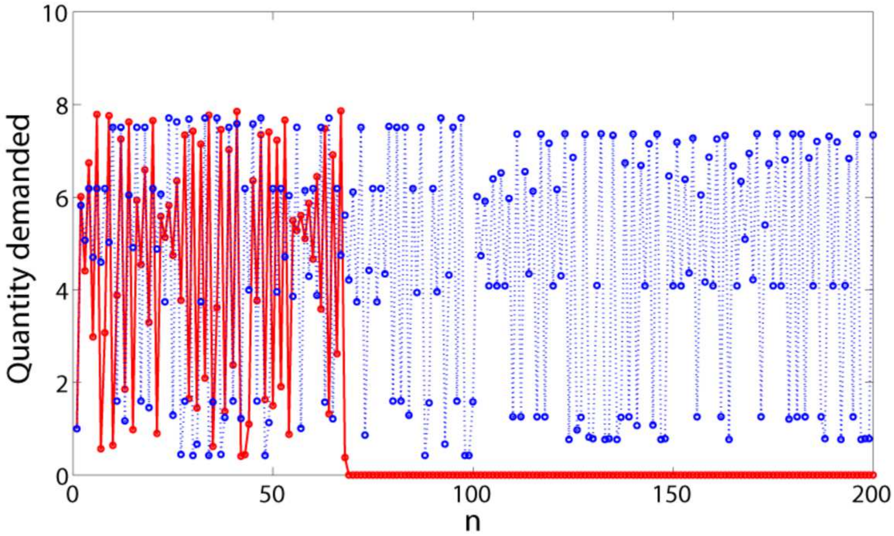

Figure 3.8. Controlled time series of the quantity demanded. Red line: time series of the quantity demanded without control exhibiting a escape towards zero, what implies an imminent market collapse. Blue dot line: controlled time series of the quantity demanded where the market collapse is avoided. This time series corresponds to 200 iterations of the system.

the expected demand the supplier knows exactly the quantity of supply. In this setup, unexpected supply fluctuations are impossible. However, there is a special case where these fluctuations can be considered feasible. In some industrial processes the supplier can only estimate the average quantity of production and not the exact amount. In this context, the partial control method can be used to control the quantity supplied and consequently the market, assuming that the firm always have some extra stock to stream into the market when positive control is needed. To demonstrate the efficiency of the partial control method in complex scenarios where the firm behavior is more sophisticated, we have chosen the cautious and optimistic supplier for this section. The model that represent this type of supplier is shown in Eqs. (3.11-3.13),

$$D_{n+1} = a - bP_{n+1}, \tag{3.11}$$

$$S_{n+1} = \sqrt{(\frac{D_n}{S_n})} \cdot S_n, \tag{3.12}$$

$$P_{n+1} = \frac{1}{1-M} \cdot \big(\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2\big). \tag{3.13}$$

We are interested in the map for the quantity supplied shown in Eq. (3.14) which is a simplification of Eqs. (3.11-3.13). The region K is defined by the interval between zero and the maximum amount of goods that can be produced using the

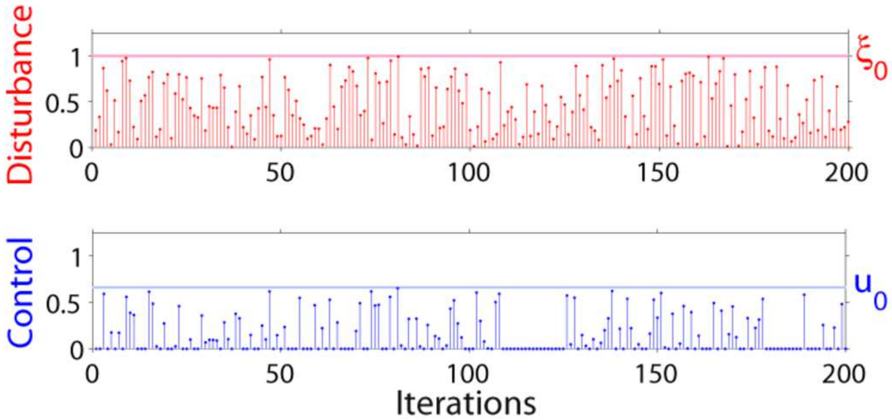

Figure 3.9. The disturbance and control applied in absolute values at each time step. Note that the control applied in order to sustain the trajectory in the transient regime is always smaller than the disturbance. The reader can check that every time step the amount of control (blue bar) is much smaller than the amount of disturbance (red bar), where the average control disturbance ratio is 0.66 again.

fix capital S_{max}. As we can see in Fig. 3.10, almost every initial supply S_0 diverges to minus infinity. Hence, independently to the quantity supplied that is chosen by the firm for S_{max}, the system will almost always lay on a transient chaos regime. We have chosen for our numerical simulations the initial region K to be the interval $S_n \in [0.15, 11.2]$ as shown in Fig. 3.10.

$$S_{n+1} = \sqrt{\frac{1}{S_n}\Big(\frac{1}{1-M} \cdot \big(a - b(\frac{F_c}{S_n} + v - vS_n + S_n^2)\big)\Big)} \times S_n. \tag{3.14}$$

Again, we have introduced a uniform noise distribution bounded by ξ_0. This disturbance can be positive when the quantity produced exceeds the expected production or it can be negative when the quantity produced is below the expected production. When positive control is needed the firm uses its extra stock to fill the shortfall. When negative control is needed the supplier destroys or simply takes out from the market the exceed quantity. As in the previous sections there is a critical value of u_0, below which no safe set exists. We have chosen for our numerical simulation $\xi_0 = 2.0$ and $u_0 = 0.33$, where u_0 is very close to the minimum value for which a safe sets exists. We evaluate the value of $f(S_n) + \xi_n$. If the point is inside a safe set, we do not apply the control. If it is outside, we relocate it inside the nearest safe point, resulting the new safe point $S_{n+1} = f(S_n) + \xi_n + u_n$. The final result of applying the Sculpting Algorithm is shown in Fig. 3.10.

The efficiency of the partial control method is shown in Fig. 3.11. We want to emphasize another powerful property that can be exploited using the partial control method. The red line in Fig. 3.11, corresponds to the time series of the quantity supplied without control. Originally, the firm can produce a limited amount of

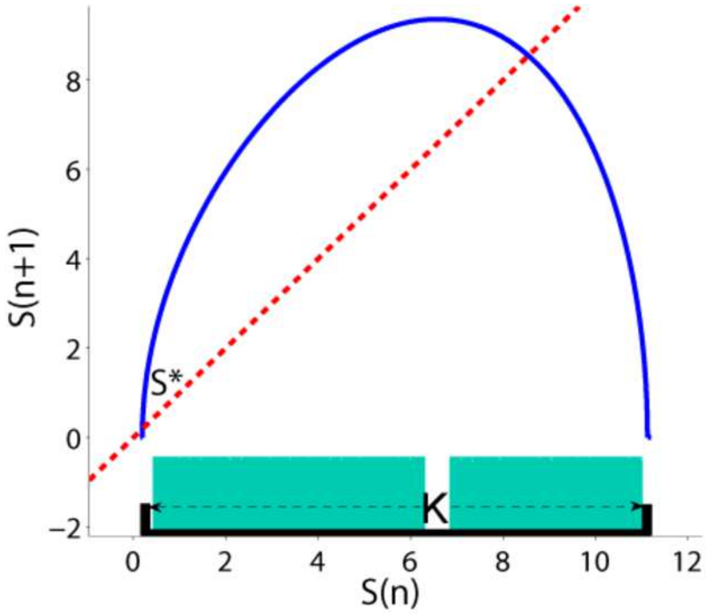

Figure 3.10. The phase space of the supply map. This figure shows the phase space of the supply map (blue line) and the intersect $S_{n+1} = S_n$ (red dash line). The region K (black thick line over the horizontal) is the region where we want to sustain the dynamics. Using the Sculpting Algorithm we have found the safe sets and we have plotted them as turquoise rectangles over the region K.

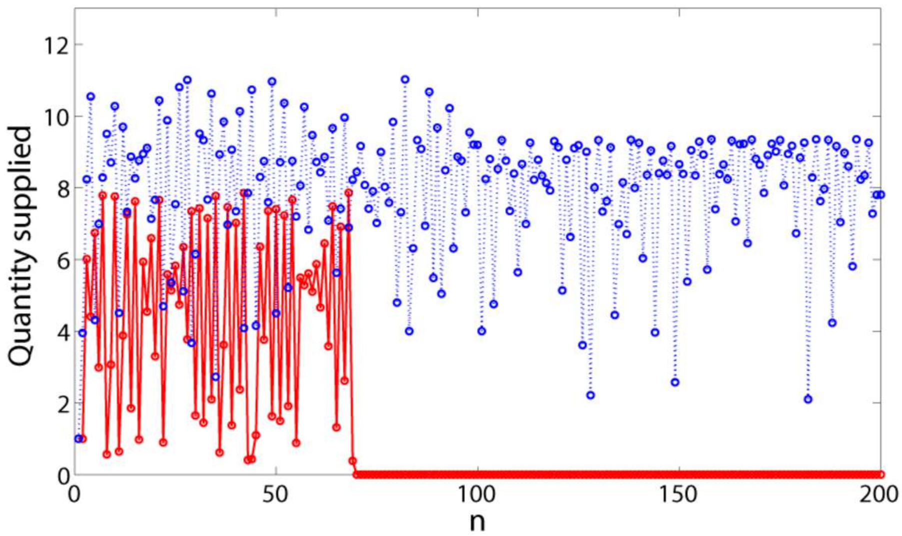

Figure 3.11. Controlled time series of the quantity supplied. Red line: time series of the quantity supplied without control exhibiting a escape towards zero, what implies an imminent market collapse. Blue dot line: controlled time series of the quantity supplied where the market collapse is avoided. This time series corresponds to 200 iterations of the system.

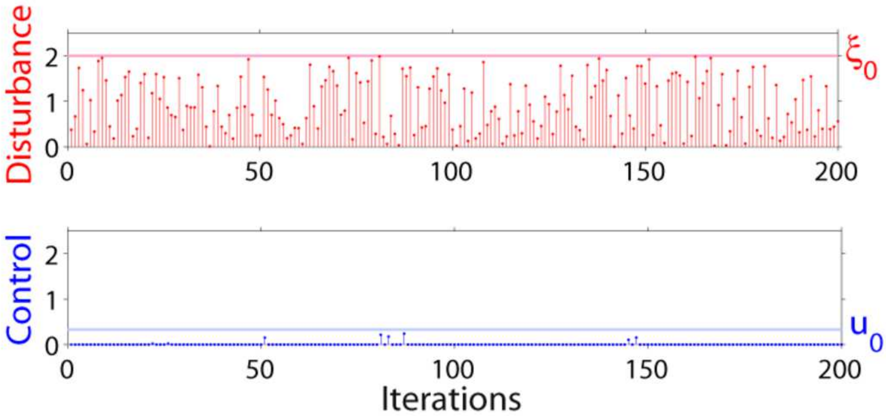

Figure 3.12. The disturbance and control applied in absolute values at each time step. Note that the control applied in order to sustain the trajectory in the transient regime is always smaller than the disturbance. The control disturbance ratio is much smaller than in the last two sections and is about 0.165.

products, in this case is around 8 products each time step. We have extended this natural barrier to 11.2 by defining a region K larger then the natural bounds of the system. The controlled trajectory is higher than the uncontrolled trajectory due to the assumption we made earlier, letting the firm to introduce positive controls using its extra stock. This interesting property [88] can be exploited by the firm. Assuming that the firm has no limited stock, it can supply all of it, without collapsing the market. Furthermore, the amount of control needed at each time step to maintain the dynamics of supply in the transient regime is extremely much smaller than the disturbances as shown in Fig. 3.12.

3.7 Conclusions

Avoiding market collapses might be a big challenge for economists. The difficulties in predicting such phenomenon due to the nonlinear interaction among the agents at the micro level, makes the engineering of a control strategy at the macro level a very hard task. This can be worst when unpredictable external disturbances are present in the system. In this chapter, we have shown that this macro state of collapse can be prevented acting on the macro level of the market when a powerful firm had influenced the quantity demanded using the partial control method. A firm with high market power might influence the demand by intervening directly in the market, buying the excess supply, or just by investing in advertising to encourage the consumption in the market. But it is more remarkable how the agents at the micro level have prevented the macro state of collapse by changing their behaviors using the partial control strategy. We have used the supply based on demand model to show how the firm can control the price trajectory, avoiding price explosion, only by changing the selling price of the product at every time step in accordance to the circumstances. We have also shown that the firm can apply the partial control strategy on the quantity supplied in some special cases. Furthermore, while it is doing that, it is able to extend the natural barriers of the system supplying more goods than before without being detrimental to the market. We have used the partial control method, that has the advantage of using a control to sustain the dynamics in the transient regime much smaller than the external disturbances introduced in the model. The pursuit for efficient control strategies to help humans dominate the economy has always been there. The wounds left by the last global crisis are a painful reminder of why we need to insist in this search. Novel control methods like the partial control method bring us closer to realize this dream.

Chapter 4

When repetition is the best strategy

"Everything existing in the universe is the fruit of chance and of necessity"

-Democritus

In our increasingly interconnected societies people must make very complex decisions in a daily manner, sometimes they must consider many correlated factors at the same time, to solve just one simple problem. Traditionally, the neoclassical economists refer to those people as *Homo economicus*, in other words, rational people with endless computing power who maximize their utility function using all the information present in the system. These *Homo economicus* agents are more similar to gods than to ordinary people. But this simplistic view has changed during the last half century thanks to the important contributions of some social scientists who focus on how ordinary people really think. These studies have changed forever our understanding on how people make decisions based on probabilities or how they perceive risk. The bounded rationality paradigm is very useful when designing social mechanisms, but it is very challenging to integrate it into economic models. Computer programs can help us to solve this problem, in particular agent based models. In this chapter using an agent based model, we study a complex decision problem where many agents need to simultaneously choose between three pools that provide some payoff depending on a stochastic function and the attendance to the pools.

4.1 Introduction

Many times, in our life, we need to make very complicated decisions that are based on a multitude of factors. For example: change our current safe job, for the chance to get hired by some better company that offer us a higher salary. In which economic asset (stock, cryptocurrency etc.) to invest. In a safe asset that has a low yield, or in a risky asset with high expected yield. What career to choose. To go or not to a public place depending on the expected attendance, to participate or not in a bet and many more. To make such decisions and develop a strategy of action, always, consciously or unconsciously, we construct one or several mental models using the data we have at our disposal. Sometimes, we look for a pattern in the data and sometimes we compute the mean or the standard deviation of an important feature in the data. Sometimes, we repeat the same decision we have made in the past or we make our decision completely at random, tossing a coin. The fact that we are not rational creatures [89, 25], and by design we are not good at computing and evaluating probabilities [24, 90], opens the door to an infinite variety of models and strategies similar to the above mentioned. Contrary to our intuition, the presence of a wide variety of strategies in a system do not necessary mean that the system will fall into a chaotic regime. The Irish economist Brian Arthur, in his famous paper on the *El Farol Bar problem* [21, 91], showed that when many strategies are present in a system, some of the global dynamics will converge towards the average optimal or desired solution of the system. This work has three goals: 1. To build a reliable ABM using the Netlogo software [92] and to simulate the proposed system [93]. 2. To compare the success of different strategies and families of strategies in a variety of situations. 3. To study the global dynamics of the system.

The structure of the chapter is as follows. Section 4.2 is devoted to the description of the problem and to define some of the key quantities in the system. The agent based model that we have developed to explore this system is described in Section 4.3. In Section 4.4, we explain in depth each one of the strategies that we have integrated in the model. Section 4.5 is dedicated to explain the methodology of the experiments that we have carried out. In Section 4.6, we show the results obtained from the different experiments. Finally, some conclusions are drawn in Section 4.7.

4.2 The problem and some definitions

Before we start, we want to define some quantities of interest in our model. The payoff P of a pool is the total amount paid by the pool at each time step. The reward R is the proportional part of the payoff that corresponds to each agent that was in that specific pool at the time of paying the payoff, that is: $R = (P/N)$, where N represents the number of agents that inhabit the pool. The parameter τ represents the agent's cost of switching between two different pools and the balance B is the total amount of rewards minus the total amount of costs, that is, $B = \sum(R) - \sum(\tau)$. In this work we assume that the goal of the agent is to maximize his balance B. To achieve this goal, the agent needs to attend the pool that offers the highest expected reward R at each time step, assuming in the case of moving to another pool that, $R \geq \tau$.

The reward R will always depend on two factors: 1. N, since the payoff is divided equally among all the agents that inhabit the pool at each time step. 2. The probability that the pool will pay its payoff P. The stable pool, at each time step, will pay a payoff that is equal to the number of agents that inhabit this pool

$P = 1\$ \times N$, that is equal to 1$ reward with a probability of 100%. This dynamic rewarding scheme reduces the risk associated with this pool to zero. In contrast to the stable pool, the payoffs paid by the low pool and high pool depend on some stochastic functions. The low pool pays 40$ with a probability of 50% at each time step and the high pool pays 80$ with a probability of 25% at each time step. Agents can decide what to do by only looking at the current state of the population and the current state of payoffs paid by the pools. In some cases the agents also have short term memory in which they can find patterns in the historical data. Considering these two factors, when the agent wants to attend the low pool or the high pool it always faces a prediction or estimation problem. The agent must predict or estimate if the pool will actually pay its payoff in the next time step, since we saw that in the high and low pools there is no absolute certainty that any payoff will be paid in the future. In addition, the agent must predict or estimate the value of N in that same pool in the next iteration, in order to compute the expected reward R.

The relationship between N and R in each pool can be written as follows: for the high pool: $R = 0.25 \times 80 \times N$. For the low pool: $R = 0.5 \times 40 \times N$ and for the stable pool: $R = (N \times 1)/N$. When the expected N in the low or in the high pool is equal or bigger than 20, there are no incentives for taking the risk in attending these pools, since the agent can get the same reward without any risk if it goes to the stable pool as shown in Fig. 4.1. We will show that when many agents are present in the system and they own a wide variety of strategies, N in the high and the low pools will approach to this *attractor* ($N = 20$), as predicted by Brian Arthur [21].

4.3 The model and observables

We have built a Netlogo model to simulate this system. The model consists of two types of agents: the Investors and the Pools. The investors attend the pools depending on their strategies at each time step. We have created 13 strategies and the user can set them up manually turning on the *Manual* switch or automatically using the *Mix* tab in the *Strategies* chooser. All investors keep track of their own R, B, the pool they attended and τ. The pools keep track of their own corresponding P, R and N. We were interested in the following observables: P, the average R of each strategy and the average R of each group of strategies, the average B of each strategy and the average B of each group of strategies and N in each pool as shown in Fig. 4.2.

4.4 Strategies

Now we can define what a strategy is. A strategy is the action taken by the agent with the goal of maximizing its balance at each time step. This action is determined by one or several models that were fitted to some data, with the goal of predicting or estimating one or both of the above mentioned factors successfully.

We have decided to distinguish between four families of strategies: *Random strategies* - where the agent attends the pools randomly. *Naïve strategies* - where

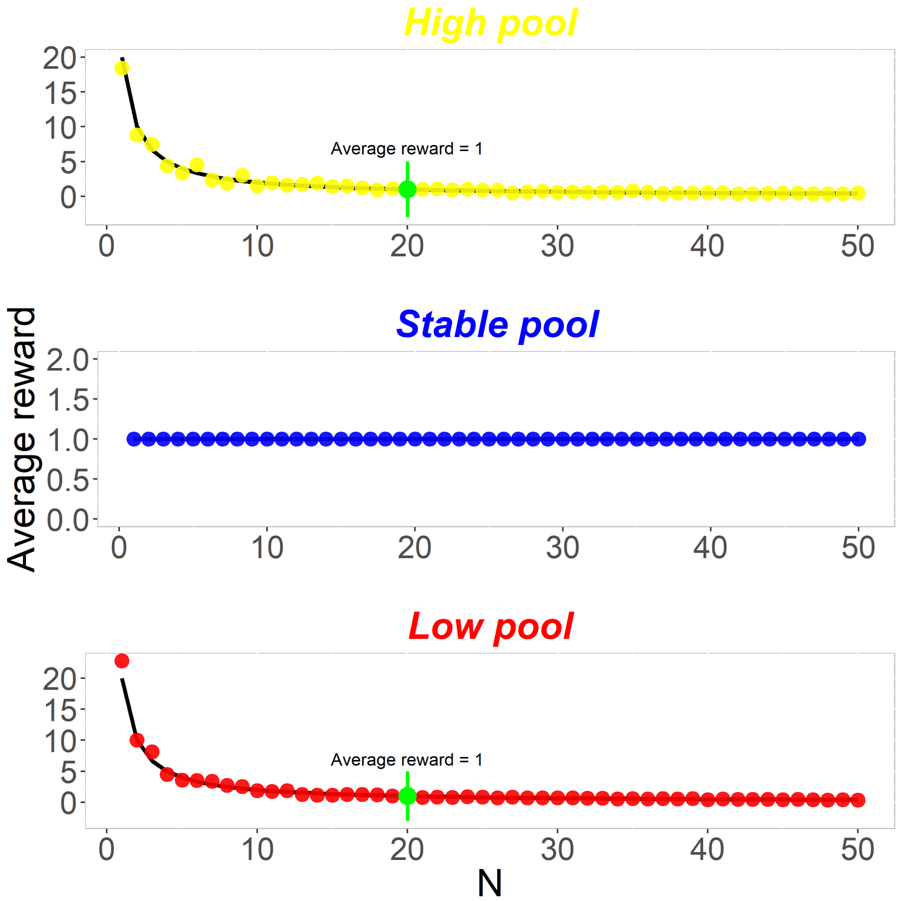

Figure 4.1. Average reward in each pool depending on its inhabitants. We have run the model 100 times for each N. We have incremented N by 1 after 100 runs. Each run we have computed the average reward for each pool and we have plotted this average (colored points) over the black lines that represent the close solutions for the functions described above, when $0 < N < 50$. We have marked N in the low and high pools (green point) when the average reward was equal to 1.

the agent repeats the same actions of previous iterations. *Aggregated strategies* - these strategies use aggregated metrics such as maxima, minima, means, modes or standard deviation of a variety of features in the data. Finally, *Pattern based strategies* - which are based on probabilities and patterns in the data. Next, we will explain some of the simple strategies we have developed for this ABM. To simplify, we have grouped the strategies in their corresponding family.

4.4.1 The random strategies family

1. *Random strategy* - at each time step, the agent randomly attends one of the pools (33.3% - stable, 33.3% - low and 33.3% - high). The agent can switch to some of the other pools, only if it has enough balance to pay the cost of τ. This

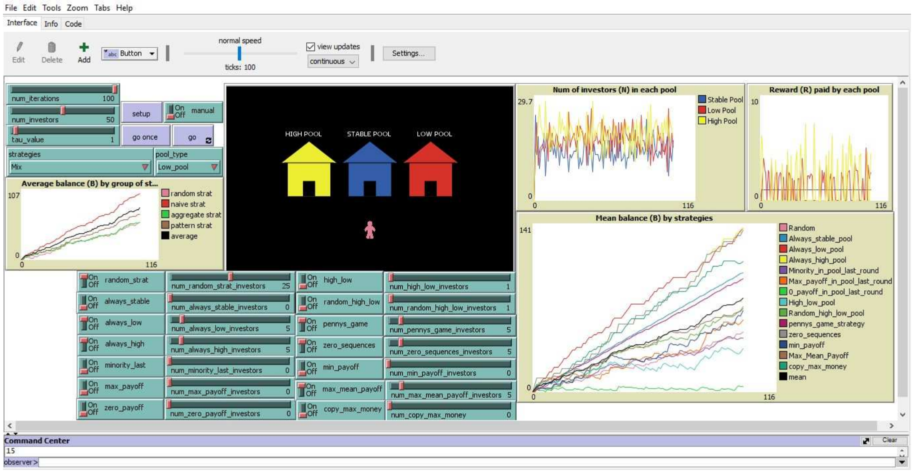

Figure 4.2. The interface of the Netlogo model. At each time step, the pink persons (Investors) move into one of the Pools (houses), depending on their strategies. This model can be set up manually letting the user experiment with any combination of agents and strategies, and it can be also set up automatically where strategies are asigned randomly, keeping the groups of investors with the same strategy proportional to other groups.

strategy has a very important role in the comparison between strategies since it allows us to compare the effectiveness of any strategy against the random case.

2. *Random high low* - exactly like the previous one with the difference that the agent randomly attends only the high or the low pool.

4.4.2 The naïve strategies family

1. *Always stable pool* - the agent always goes to the stable pool.

2. *Always low pool* - the agent always goes to the low pool.

3. *Always high pool* - the agent always goes to the high pool.

4. *High low* - In this strategy the agent attends only the high or the low pool. At the first time step the agent randomly attends to the high or the low pool. When it has acomulated enough balance, it starts to switch each time step between its first choice and the other pool.

4.4.3 The aggregated strategies family

1. *Minority in the previous round* - at the first time step, the agent randomly attends one of the pools. In consecutive time steps the agent goes to the least

populated pool of the previous iteration. If its movement imply switching to some of the other pools, the agent can do so only when it has enough balance. If there are two pools with the same minimum N, the agent chooses randomly between those pools.

2. *Maximum reward in the previous round* - at the first time step, the agent randomly attends one of the pools. In consecutive time steps the agent goes to the highest rewarded pool of the previous round. If its movement imply switching to some of the other pools, the agent can do so only when it has enough balance. If there are two pools with the same maximum R, the agent chooses randomly between those pools.

3. *Minimum reward in the previous round* - at the first time step, the agent randomly attends one of the pools. In consecutive time steps the agent goes to the lowest rewarded pool of the previous iteration hoping to get high reward in the next round. If its movement imply switching to some of the other pools, it can do so only when he has enough balance. If there are two pools with the same minimum R, the agent chooses randomly between those pools.

4. *Maximum average payoff in the previous round* - at the first time step, the agent randomly attends one of the pools. In consecutive time steps the agent goes to the pool that paid the highest average payoff in the previous time step. The average payoff is simply the total payoff paid in each pool divided by the number of time steps. If the agent movement imply switching to some of the other pools, it can do so only when it has enough balance. If there are two pools with the same average P, the agent chooses randomly between those pools.

4.4.4 The pattern based strategies family

1. *Payoff zero in the previous round* - at the first time step, the agent randomly attends one of the pools. When the agent has enough balance it moves to the pool that paid zero payoff in the last time step with the hope of getting some reward. If there are two pools with zero payoff, the agent makes a random choise between those pools.

2. *The Penny's game strategy* - this strategy is based on Walter Penny solution to the problem of predicting random binary sequences [94, 95]. There are higher probabilities of obtaining a certain binary sequence after a given initial binary sequence. This strategy tries to exploit this curious solution. The agent always starts in the stable pool, and after three iterations, depending on the pattern shown in Table 4.1, the agent decides to move to the low pool or to stay in the stable pool and so on.

3. *Sequences of zeros* - The idea behind this strategy is the following: The high pool has much lower probability of paying the payoff than the low pool. This

behavior gives rise to very long sequences of payoff zero (in red), as shown below in the simulated sequence of the first 100 payoffs paid by the high pool. Assuming that $\tau = 1$, the agent takes advantage of these long sequences of zeros, moving for at least three iterations to the stable pool and earning an additional 1$ in respect to the *Always high pool strategy*. The agent always starts in the high pool, and every 4 iterations it evaluates the sequence of the last 4 payoffs paid by the high pool. When it comes across the sequence: $[bbb0]$, where, b is some number bigger than zero (in green), the agent moves to the stable pool and stays there for 4 iterations, then it returns to the high pool. If the high pool did not pay a positive payoff in these 4 iterations, then the agent earns additional 2$. In the example below, the agent has an advantage of 4$ over an agent who would always stay in the high pool.

0 0 0 0 80 80 80 0 0 0 0 0 80 0 0 0 80 80 0 0 80 0 80 80 0 0 0 0 80 80 0 80 0 0 80 0 0 0 0 0 0 80 0 80 0 0 0 80 0 0 0 0 80 0 80 0 0 0 0 80 0 80 80 80 80 0 0 0 0 0 0 0 80 80 0 80 0 0 0 0 0 80 0 0 0 0 0 0 0 80 0 80 0 0 0 0 80 0 0 0

Table 4.1. This table represents the 8 possible $3-bit$ sequences of payoffs in the low pool and the most probable sequences to occur after them. In the last column the corresponding action to each initial sequence of payoffs is displayed.

Initial sequence	Most probable sequence	Odds in favour of the probable sequence	Action
0, 0, 0	40, 0, 0	7 - 1	Move to low pool
0, 0, 40	40, 0, 0	3 - 1	Move to low pool
0, 40, 0	0, 0, 40	2 - 1	Stay in stable pool
0, 40, 40	0, 0, 40	2 - 1	Stay in stable pool
40, 0, 0	40, 40, 0	2 - 1	Move to low pool
40, 0, 40	40, 40, 0	2 - 1	Move to low pool
40, 40, 0	0, 40, 40	3 -1	Stay in stable pool
40, 40, 40	0, 40, 40	7 - 1	Stay in stable pool

4.5 Methodology

To get more insight about different aspects of this system, we have designed four experiments. In each experiment we have varied some key parameter to understand its influence on the system. We have repeated each experiment multiple times to compute aggregated statistics and to be sure that our results are consistent. In this section we will briefly introduce each experiment and in the next section we will dive into the results.

1. *The mix strategies experiment* - the goal of this experiment is to study which strategy generates the highest balance B on average, when 50 investors ($N =$

50) are present in the system assuming that $\tau = 1$. In this set up, the investors form small groups of $2 - 5$ investors. Each group uses one of the strategies that we have introduced in the previous section. In this experiment we were also interested in the attendance dynamics to the different pools, or how N in each pool changes over time. We have simulated the system for 100 iterations and we have repeated this simulation 100 times.

2. *The influence of τ experiment* - the goal of this experiment is to study the influence of τ on the average balance generated by each strategy and the attendance to the pools. We have increased τ systematically and we have measured all the observables. We have simulated the system for 1000 iterations and we have repeated this simulation increasing the value of τ by 0.1 inside the subset $\tau[0.1, 5]$.

3. *The Payoff experiment* - the goal of this experiment is to study the influence of the amount (payoff) paid by the high pool and the stable pool on the average balance of each strategy. We have increased the payoff of the high pool by 2 and the payoff in the stable pool by 0.1 on each experiment and we have measured all the observables of interest. We have run the model for 100 iterations and we have computed the average balance of each strategy in each round. Then we have repeated this simulation 100 times and we have computed the mean average balance of each strategy in all rounds. We have repeated this experiment for each payoff paid by the high pool inside the subset $P[40, 120]$ and for each payoff paid by the stable pool inside the subset $P[0, 10]$.

4. *The amount of investors experiment* - the objective of this experiment is to study the influence of the number of investors present in the system on the average balance of each strategy and the attendance dynamics of the pools. We have increased the number of investors N by 2 on each experiment and we have measured the observables of interest. We have run the model for 100 iterations and we have saved the average balance of each strategy and the attendance to each pool. Then we have repeated this simulation 100 times and we computed the mean attendance to each pool and the mean average balance of each strategy. We have repeated this experiment for all N inside the subset $N[20, 100]$.

4.6 Results

1. *The mix strategies experiment* - a typical time series generated by this simulation is shown in Fig. 4.3. Three clusters of strategies emerged after only 50 iterations. A big cluster of strategies evolve close to the mean (black line) in the middle of Fig. 4.3. A cluster of strategies that did better than the mean is located above the middle cluster and a cluster of strategies that did worse than the mean is located in the bottom of the figure. There are strategies that did better than others. For example, the *Always the low pool* strategy (red

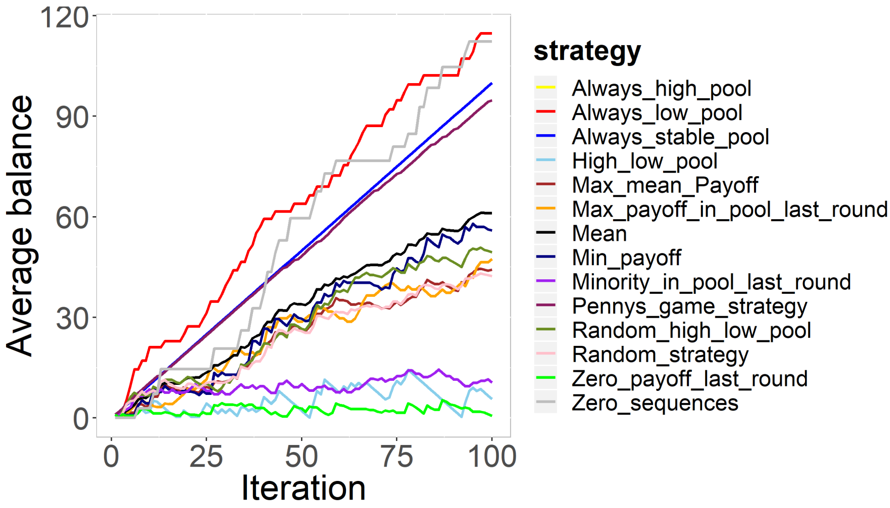

Figure 4.3. The average balance of each strategy at each time step. At each iteration we have computed the average balance of all the agents that share the same strategy. The black line, represents the mean balance of all strategies. The strategies represented by the following colors: red, blue, grey, yellow, olive and purple, were much successful (higher) than the mean after 100 iterations.

line) or the *Always stable pool* strategy (blue line) have got very high balance on average. An interesting question is if these results will persist, if we repeat this experiment multiple times and for longer time periods.

To answer these questions, we have started by collecting the closing balances of all the investors in our model during the first five simulations. Then we have grouped the investors by their correspondent strategies, and then, we have plotted the distributions of the closing balance of all strategies, as shown in the box plots in Fig. 4.3. As shown in Fig. 4.3, the three most successful strategies were *Always low pool*, *Always high pool* and *Zero sequences*. These three strategies belong to different families of strategies. To analyze the average balance of each family of strategies, we have run 100 simulations for 100 iterations each one. Each time step we have computed the average balance of all the investors that belong to each family of strategies as shown in Fig. 4.5. Finally, we have collected the average balance on the last iteration (100) of each strategy and we have computed the corresponding density functions. We have plotted the density of each strategy within its corresponding familiy to ilustrate the diviations between different strategies inside each family, as shown in Figs. 4.6-4.9.

The strategies that belong to the naïve family present the highest means. The *Always low pool* strategy (red density plot in Fig. 4.7) is less skewed than the *Always high pool* strategy, in terms of mean balance. The low pool pays higher rewards in average in a steady manner, which leads to high balance

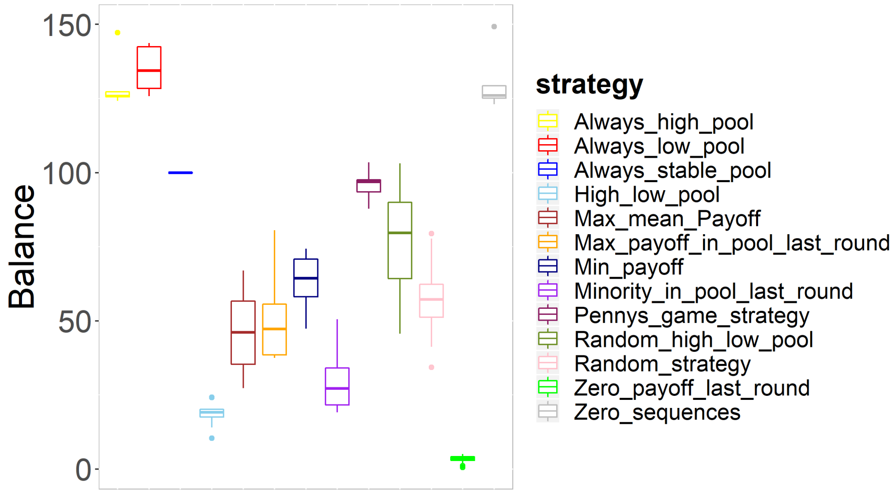

Figure 4.4. The distribution of the closing balance of each strategy during the first five simulations.

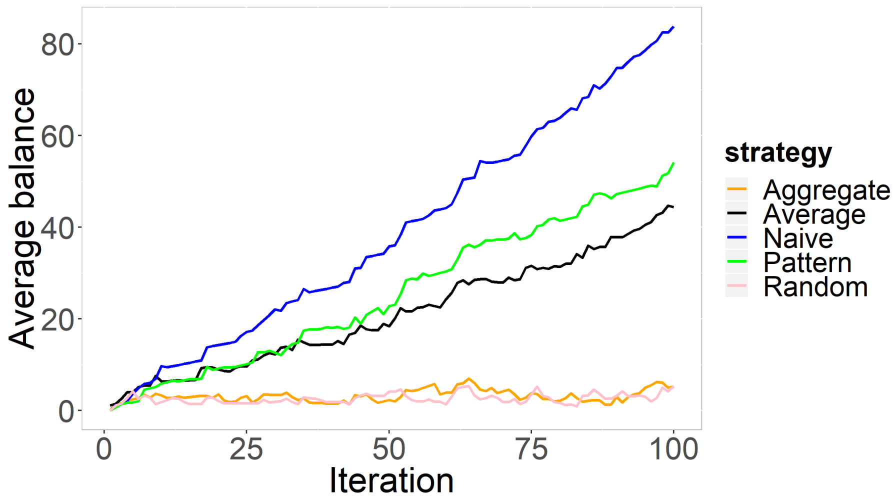

Figure 4.5. The average balance by families of strategies. At each iteration we have computed the average balance of all the agents that belong to the same family of strategies. The black line represents the average balance among all families. The strategies represented by the blue and the green lines (Naïve, Pattern based) did better than the average. This two families are much more successful than the other two families.

that is much closer to the mean, compared to the high pool. We believe that the principal causes of this distribution is the relative high probabilty of paying a payoff and the conservative behavior of the investors who bet on this strategy. The *Zero sequences* strategy has also a very high mean balance, but in contrast to the naïve strategies it has a very skewed density, much more than the *Always high pool* strategy. This observation means that the *Zero*

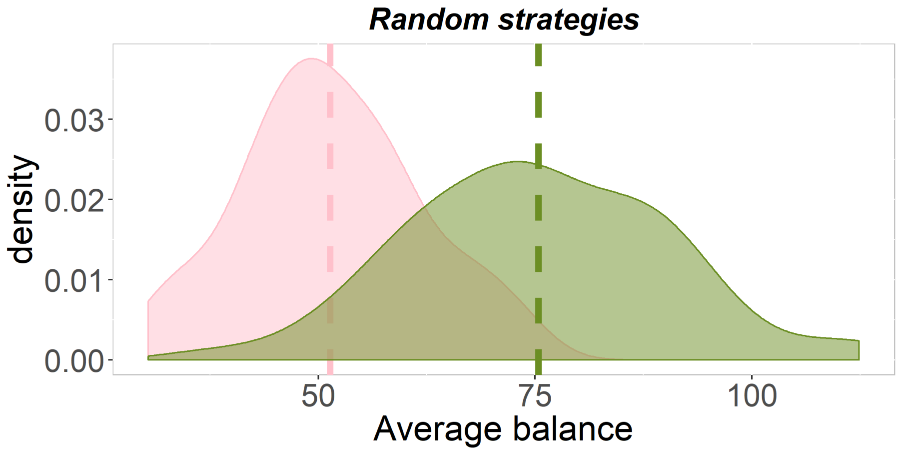

Figure 4.6. The density functions of the strategies belonging to the random family. The density functions of the *Random strategy* (pink) and the *Random high low strategy* (olivegreen). The dash line represents the center of mass of each density function.

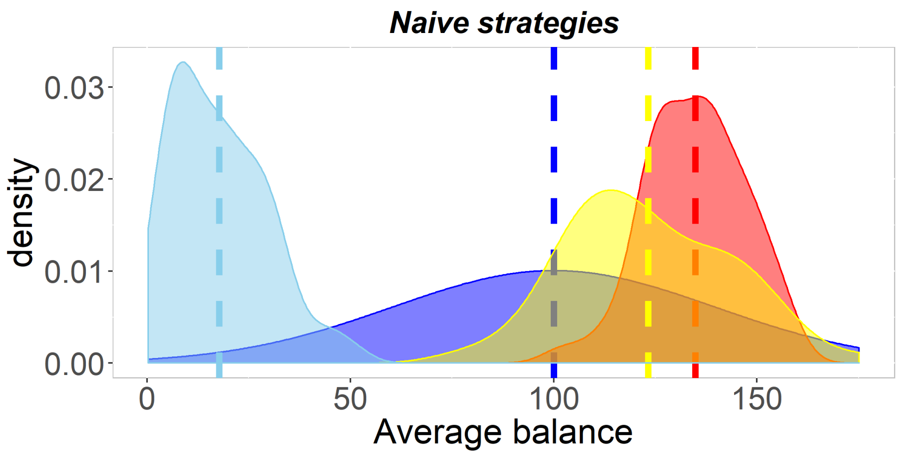

Figure 4.7. The density functions of the strategies belonging to the Naïve family. The density functions of the *Always stable pool* strategy (blue), *Always high pool* strategy (yellow), *Always low pool* strategy (red) and the *High low strategy* (sky). The dash line represents the center of mass of each density function.

sequences strategy is much more riskier strategy compared to the *Always high pool* strategy. In some cases the *Zero sequences* strategy did much better than the naïve strategies, but as shown in Fig. 4.4, this cases are rare thus we can refer to them as outliers.

In this experiment we were also interested in the dynamics of N. We show in Fig. 4.10 how N in each pool evolved over the first 100 iterations when all strategies are being used. We can clearly see how the attendance is getting close to the *attractors* discussed in previous sections as shown in Fig. 4.1. The

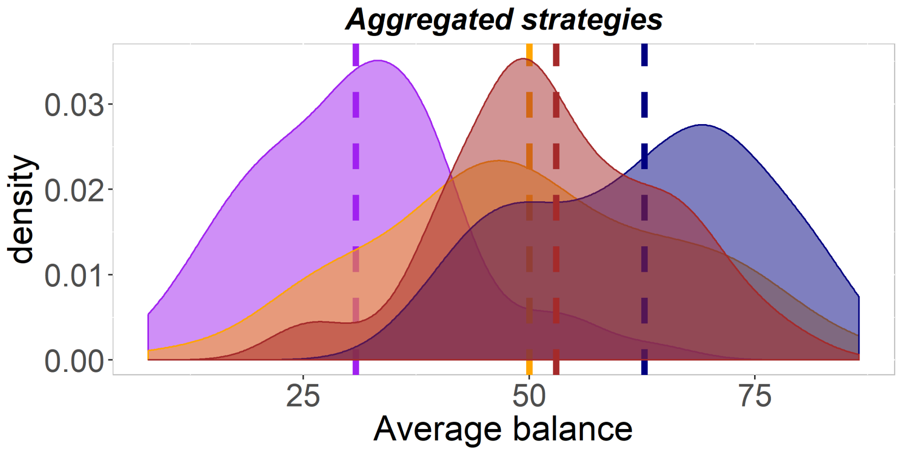

Figure 4.8. The density functions of the strategies belonging to the aggregated family. The density function of the *Minority in the previous round* strategy (purple), *Maximum reward in the previous round* strategy (orange), *Minimum reward in the previous round* strategy (blue) and *Maximum average payoff* strategy (brown). The dash line represents the center of mass of each density function.

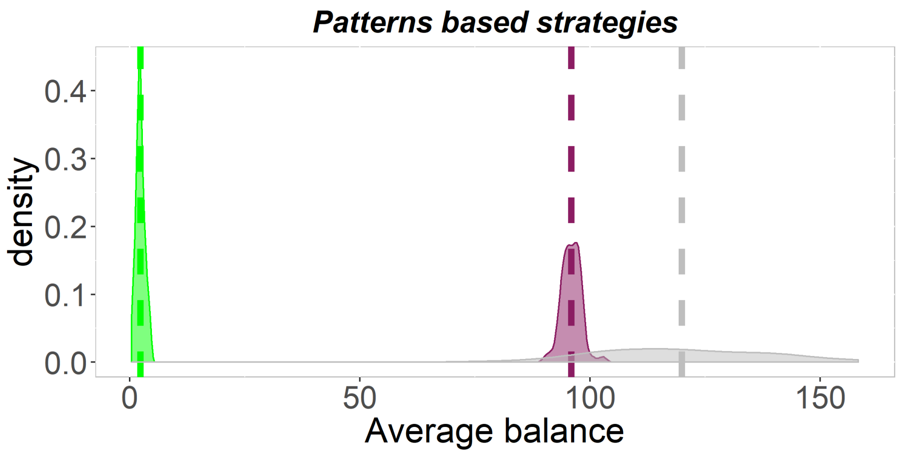

Figure 4.9. The density functions of the strategies belonging to the pattern based family. The density function of the *Zero payoff in the last round* strategy (green), *Penny's game strategy* (purple) and *Zero sequences* strategy (grey). The dash line represents the center of mass of each density function.

high pool attendence is near 20 agents on average, the attendance to the low pool is a little smaller, but not too far from the 20 attractor, and the attendance to the stable pool is near 10 agents. These findings show that when agents face a complex strategic decision making situations (much more complex than Arthur's El Farol original problem), if there are many strategies involved, we will observe the same attendence dynamics emphasized by Arthur in his famuos

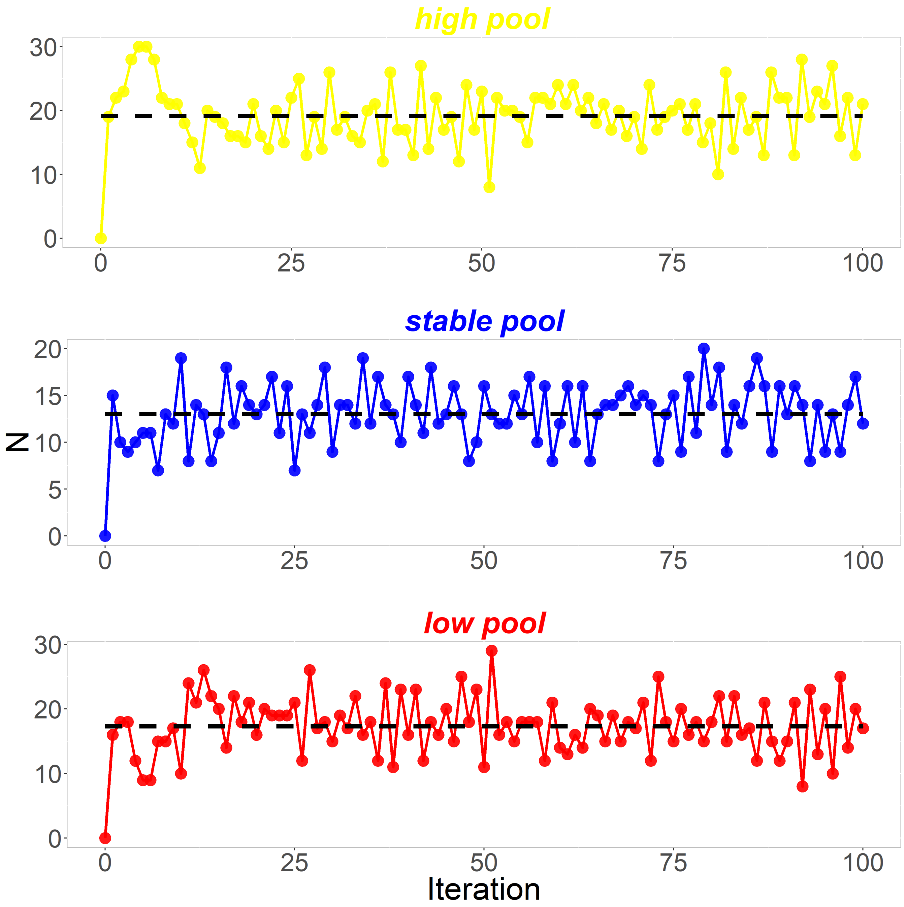

Figure 4.10. Time series of the attendence to the different pools when we run a mix strategies simulation. This figure shows the evolution of N in each pool at each time step in the first 100 iterations of the simulation. The black dashed lines represent the average attendance to each pool in the first 100 iterations. The average attendance to the high pool, low pool and stable pool is: 19.16, 17.31 and 13.01 investors respectively.

paper. Note that only six from the thirteen strategies involve intrinsically the attendance to the low or high pools. Another important observation is that the convergence to those attractors is very fast. It takes about 10 iterations to converge.

2. *The influence of* τ *experiment* - the cost of changing strategy, that is τ, has a huge effect mainly on the average balance of the strategies that belong to the Aggregate and the Random families as shown in Fig. 4.11. When $\tau = 3$, almost all strategies breakdown. It becomes much safer to stay in some pool and do not move from it. The surprising result is that the *Always low pool* strategy seems to be benefited from the increment of τ, since the average balance of the investors that have used this strategy is increasing relative to all the others. Another surprising result that can explain, why we see increments in the mean

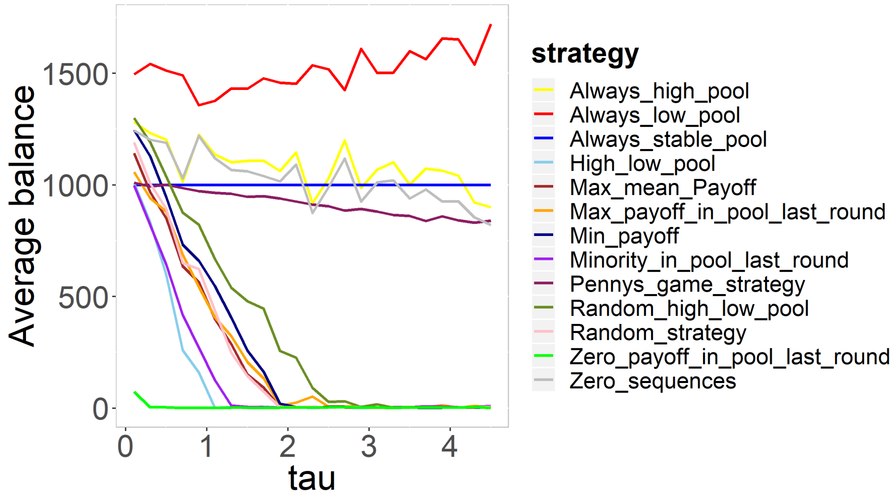

Figure 4.11. This figure shows the impact of τ on the average balance of each strategy. We have run the model for 1000 iterations saving the final average balance of each strategy in each repetition. We have increased τ by 0.1 in each simulation we have plotted the average balance of each strategy vs the corresponding value of τ.

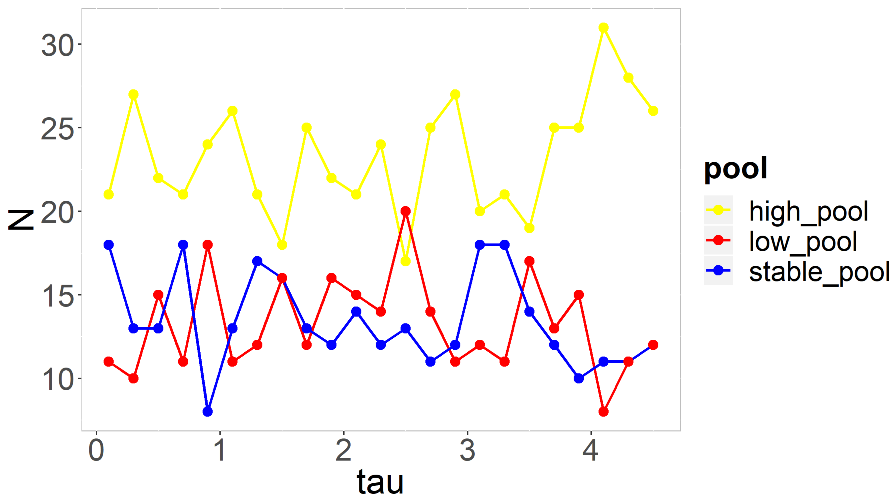

Figure 4.12. This figure shows the impact of τ on the attendance to each pool. We have run the model for 1000 iterations saving the attendence to each pool in the last iteration. We have increased τ by 0.1 in each simulation and we have plotted the final attendence to each pool vs the corresponding value of τ.

balance of investors with *Always low pool* strategy, is that the increment in τ seems to increase the attendance to the high pool as shown in Fig. 4.12.

3. *The payoff experiment* - strategies in which the agents attend the high pool frequently such as *Always high pool* or *Zero sequences* are influenced by the

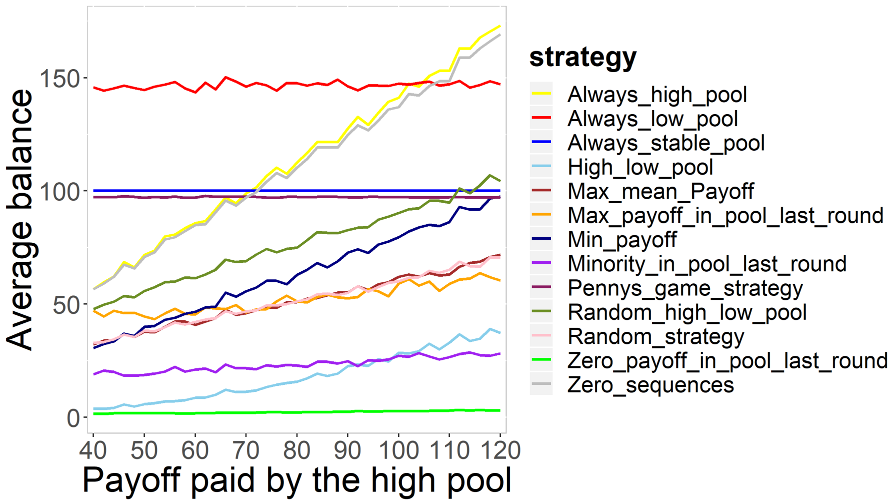

Figure 4.13. This figure shows the impact of the amount of payoff paid by the high pool on the average balance of each strategy. Naturally, all the strategies in which the investors visit in some frequency the high pool are positively influenced by the increment of P paid by the high pool. The *Always high pool* and the *Zero sequences* strategies enjoy the most from this increment, however only when the payoff is bigger than 105$ they actualy do better than the *Always low pool* strategy. This result reaffirms the advantage of the *Always low pool* strategy over the others.

increments of the payoff paid by this pool as shown in Fig. 4.13. Surprisingly the *Min payoff* strategy is also heavily influenced by those increments. For the stable pool we can appreciate how only the *Penny's game*, *Always stable pool* and *Max payoff in pool last round* strategies benefit from these increments in the payoff paid by the stable pool as shown in Fig. 4.14.

4. *The amount of investors experiment* - as expected, we have found a negative effect on the average balance of the different strategies when we increment the number of investors present in the system. The *Always stable pool* and *Penny's game* strategies are less sensitive to this increment of N as shown in Fig. 4.15. They are the best strategies even when 70 investors are present in the system. There is a critical number of investors ($N = 73$) in which the *Always stable pool* becomes the best strategy to follow. This is meaningful, since we can clearly see in Fig. 4.15, where each strategy breaksdown relative to the most safer strategy *Always stable pool*. Finally, Fig. 4.16 shows how the increment of N has a positive effect on the attendence to the high pool.

4.7 Conclusions

In this work we have studied the system proposed by the Complexity Challenge Team from two directions, the ecosystem of strategies and the global dynamics of the system. We have developed thirteen strategies, some of them very simple and

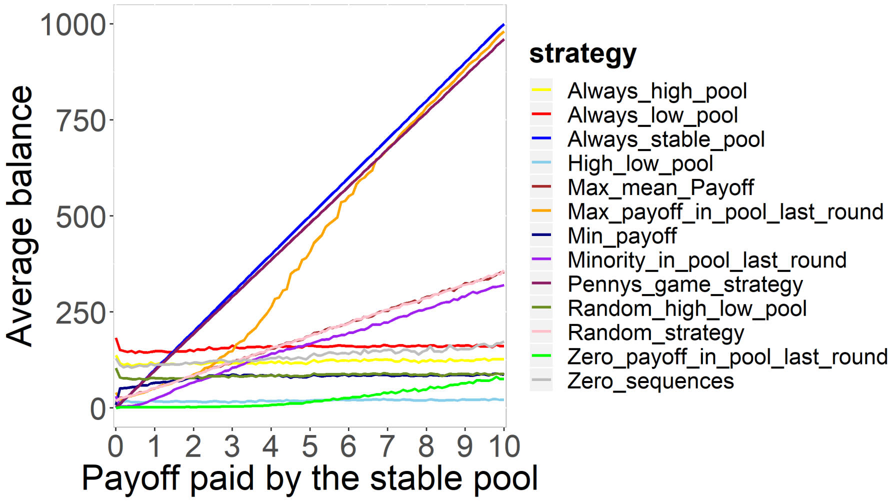

Figure 4.14. This figure shows the impact of the payoff amount paid by the stable pool on the average balance of each strategy. Obviously small increment in P paid by the stable pool makes it safer and more profitable. As shown in the figure above, the *Always stable pool* strategy becomes the most profitable strategy when the payoff paid by this pool is bigger than 1.6$.

others more complex. Then we have grouped all the strategies in four families of strategies. We simulate the system using our Netlogo model, and we have find that the most successful families are the naïve and the Pattern based families. This means that in this system when the investors use many different strategies, decisions that are made based on patterns in the data or repetition lead to higher balance on average. We have learned that the most powerful strategies are the simplest naïve strategies. The *Always low pool* is the best strategy even when τ increases. The *Always stable pool* strategy becomes the best when N increases over 73 agents. Although we have simulated a small number of strategies (13 strategies), we have shown that the average N in each pool approaches the attractor of $N = 20$ in the low and in the high pool. This result is surprising because we have thought that much more strategies are needed to observe this dynamics. In future studies we want to develop strategies based on machine learning and to study deeper the basic patterns of the *Zero sequences* strategy.

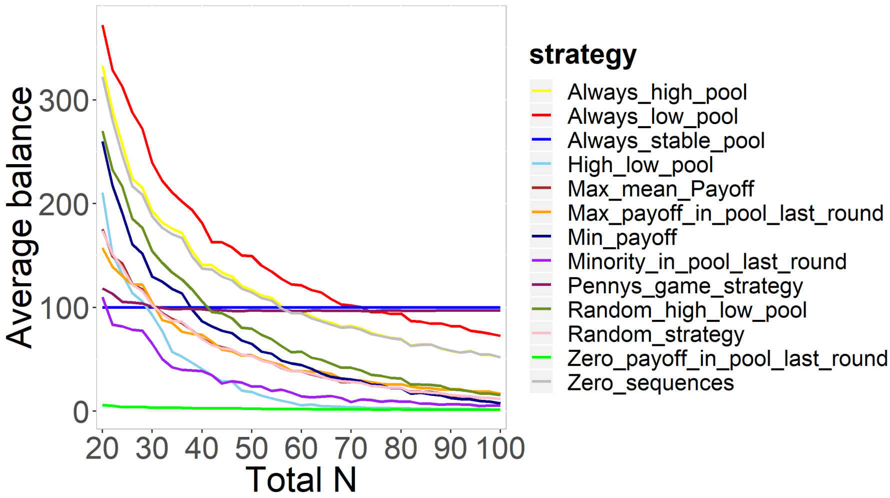

Figure 4.15. This figure shows the impact of the amount of investors present in the system on the average balance of all strategies. Logically the amount of investors present in the system has negative effect on their total balances. An interesting discovery is the resistence of the *Penny's game* strategy to this incrementation of investors.

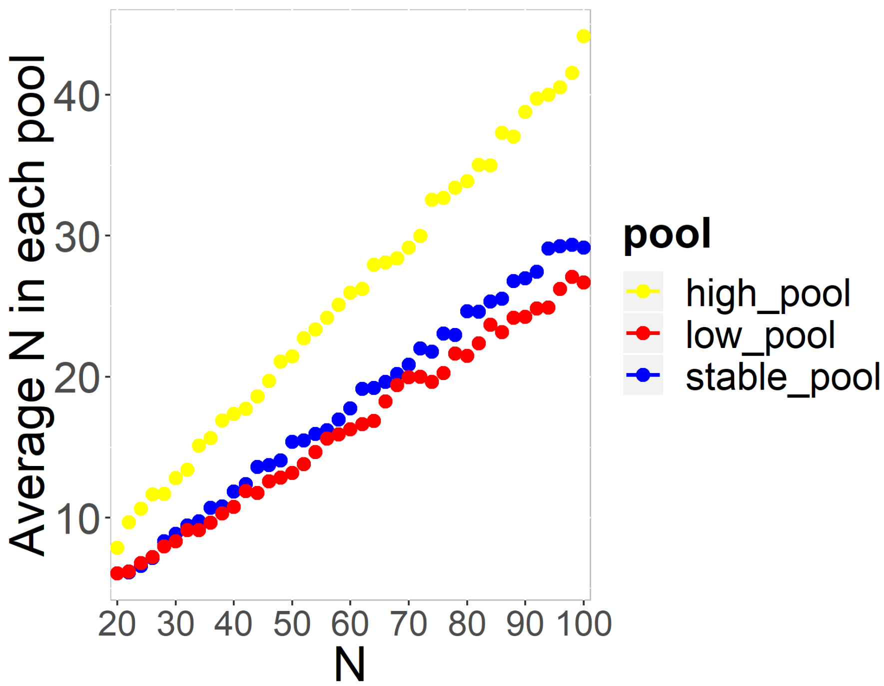

Figure 4.16. This figure shows the impact of the total N in the system on the average attendance to each pool. Surprisingly, the high pool tends to recieve more investors on average.

Chapter 5

Making Predictions on Fractal Basins

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

-Albert Einstein

The Duffing oscillator is one of the best-known models of nonlinear oscillators, with applications in many fields of applied sciences and engineering. It is frequently used to model nonlinear springs, dissipative systems or the market dynamics. This system exhibits a fractal phase space for some parameter values. In some cases this fractal structures verify the Wada property. A Wada basin is essentially, a basin that simultaneously shares its fractal boundary with two or more basins of attraction.

The immediate consequence of the Wada property is the intrinsic difficulty to make predictions, since an initial condition located in the Wada boundary can potentially evolve towards any of the attractors presented in the phase space. This is of a great importance, since we are used to classic determinism ideas, where once we fix the initial condition, automatically we know the exact evolution of the orbits. From an experimental point of view, fixing an initial condition with an infinite precision is not possible, from what it arises to a serious problem of predictability in physical and economic systems. In this chapter we will study some of the statistical properties that appear in this scenario.

5.1 Introduction

Predicting the future state of a nonlinear dynamical system may be very challenging. Recently the use of sophisticated prediction techniques, like neural networks, has allowed researchers to improve the prediction ability in such systems [96]. But this type of methods cannot be always easily applied. In many nonlinear dynamical systems, complex structures arise and change their shape within phase space as

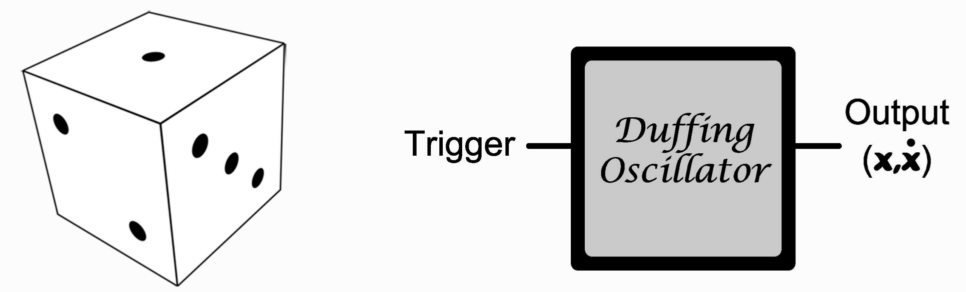

Figure 5.1. Black box diagram. We consider that the dynamical system that we are going to study is a black box to which we do not have any internal access. We can only measure the final state of the system for a given initial condition. In this sense, the problem that we face is very similar to the problem of predicting the final state of a die.

one parameter is varied. Basins of attraction are an interesting example of these structures in dissipative and Hamiltonian systems. Roughly speaking, we can say that a basin of attraction is the set of initial conditions that evolve in time towards a given attractor. In many nonlinear systems there are several attractors coexisting in phase space, which can have fractal boundaries separating their basins. This fact can make the study of the global dynamics and the predictability of the system a very difficult task. Nonlinear systems with fractal basins can be classified basically in four different categories: intertwinned basins, Wada basins, riddled basins and sporadically fractal basins [97]. When a dynamical system possesses this kind of basins it is very difficult to make predictions, due to the fact that there is an intrinsic uncertainty on the final state of a given initial condition taken in the neighborhood of the fractal boundary. The physical reason behind this is the finite accuracy in the measurement of the initial conditions for any real system. Furthermore, in systems with fractal basins there are infinitely many close points that can go to a different attractor. The situation gets even more complicated if we do not have access to the time series of the dynamical system and the only observables of the system are the attractors.

Although the problem is far from being solved, recently two useful ideas proposed by Menck *et al.* and Daza *et al.* namely basin stability [98] and basin entropy [99] have shed some light on important properties of complex basin structures. Here, we present a general procedure to provide some kind of statistical prediction in nonlinear systems with fractal basins, where the only observables that we have access to are the attractors of the system. We assume that we are not able to measure the time series before they reach the final attractor, but we assume that we have some knowledge about the probability density function of the initial conditions. In this way, we consider that the dynamical system is like a black box, as depicted in Fig. 5.1, where only the final output can be measured. In this framework, the behavior of the dynamical system is very similar to that of a die, although the behavior of this one is neither chaotic nor random [100]. The key point of the prediction mechanism developed here, is (as we show in several ways) that the ratio or probability of initial conditions going to each attractor in the phase space is scale free. This is precisely what allows the statistical prediction. We show here how this procedure works for Wada basins, but it should also work for systems showing any of the other kind of fractal basins.

A dynamical system has Wada basins if it has three or more basins sharing the same fractal boundary. This topological idea was introduced by Kennedy and Yorke [101]. Wada basins usually appear in two-dimensional dynamical systems as a result of a boundary crisis of a chaotic attractor. This fact often leads to the fractalization of the entire basin boundary. Wada basin boundaries are frequently observed in both dissipative and Hamiltonian systems. We can find this topological property in relation to mechanical models of billiard [102] or chaotic advection of fluid flows [103] and in the context of the Hénon-Heiles Hamiltonian system in celestial mechanics [104]. Due to the structural complexity of the Wada basin boundaries, in practice, these structures imply serious problems in the long term prediction of dynamical systems, also known as final state sensitivity [105, 106].

Here, we study the Duffing oscillator for a choice of parameters that verifies the Wada property, based on the work of Aguirre and Sanjuan [107]. The Duffing oscillator is one of the best known models of nonlinear oscillators, with applications in many fields of applied sciences and engineering. The structure of the chapter is as follows. Section 5.2 is devoted to the description of the Duffing oscillator and the methodology used to explore its phase space. The one-dimensional analysis of the model is described in Section 5.3. The two-dimensional analysis is done in Section 5.4. The implication of fractal boundaries on the probabilities in ending up in each basin of attraction is given in Section 5.5. Finally, some conclusions are drawn in the last section.

5.2 The Duffing oscillator

We consider here the periodically driven Duffing oscillator [111] that is described by the following differential equation (1),

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\cos(\omega t). \tag{5.1}$$

The Duffing oscillator of Eq. (5.1) has a transient chaotic behavior and there are three coexisting periodic attractors whose basins of attraction have Wada boundaries[1] [107]. We have used the stroboscopic map (with $T = 2\pi$) associated to the

[1]It is said that x belongs to a particular boundary when there is an open set surrounding x

Duffing oscillator to compute the position of the attractors in phase space. We define as $P1R$ and $P1L$ the period-1 attractors located on the right and on the left, respectively. We define as $P3L$, $P3C$ and $P3R$ the points belonging to the period-3 attractor. The period-1 attractors are located at $P1R \approx (0.815, 0.242)$ and $P1L \approx (-0.933, 0.299)$. The period-3 attractor is located at $P3L \approx (-1.412, -0.137)$, $P3C \approx (-0.354, -0.614)$ and $P3R \approx (0.645, -0.464)$ [107]. The frequency is a critical parameter in the study of nonlinear oscillators [108, 109]. But this parameter stops being so important in chaotic systems, since they have a broad spectrum systems which covers a wide range of frequencies [110].

To compute the basins of attraction, we have taken all the initial conditions inside the square $[-2, 2] \times [-2, 2]$ of the phase space, and we have integrated the system using a fourth-order Runge-Kutta integrator with a fix integration step of $2\pi/4 \times 10^5$, until their orbits reach the corresponding attractor. Different colors have been chosen according to which attractor an initial condition goes to, as shown in Fig. 5.2. Every initial condition belonging to the basins of attraction of the period-1 attractors $P1L$ and $P1R$ have been plotted in red and green respectively. All the initial conditions belonging to the basin of attraction of the period-3 attractor have been plotted in blue. The phase space resolution depends on the amount of points taken in the horizontal and vertical axes. More points imply more resolution. For example, if we divide $\dot{x}$ and x in 400, we obtain a 400×400 matrix with $160,000$ initial conditions, where every initial condition has a two decimal precision. In the next sections we have studied the following matrices of initial conditions: 50×50, 100×100, 200×200, 300×300, 400×400, 500×500, 1000×1000, 2000×2000 and 3000×3000. We have started with a matrix of $2,500$ (50×50) initial conditions and finished with a matrix of 9×10^6 (3000×3000) initial conditions.

To find the probability of reaching a given state in a dynamical system it is necessary to know its final probability density function (invariant measure). The evolution of an arbitrary probability density function in a dynamical system f is described by the Perron-Frobenius operator.

$$\rho_{n+1}(x) = \mathfrak{L}_{PF}\rho_n(x), \tag{5.2}$$

where ρ_n is the natural invariant after the $n-th$ iteration of the map. The operator can be explicitly written as,

$$\mathfrak{L}_{PF}\rho_n(x) = \int \rho_n(x)\delta(x - f(y))dy. \tag{5.3}$$

When only a finite number of non-chaotic attractors can be found in phase space, the evolution of the probability density function described by the Perron-Frobenius operator converges to delta functions. With knowledge of its invariant measure it is possible to determine the probability of ending on each attractor. However, in

which intersects two basins. The basin boundary is the collection of all the boundary points and is an invariant set. The basin boundary may be a smooth curve, but it can also have a fractal structure. A basin boundary is fractal if it contains a transversal homoclinic point [113].

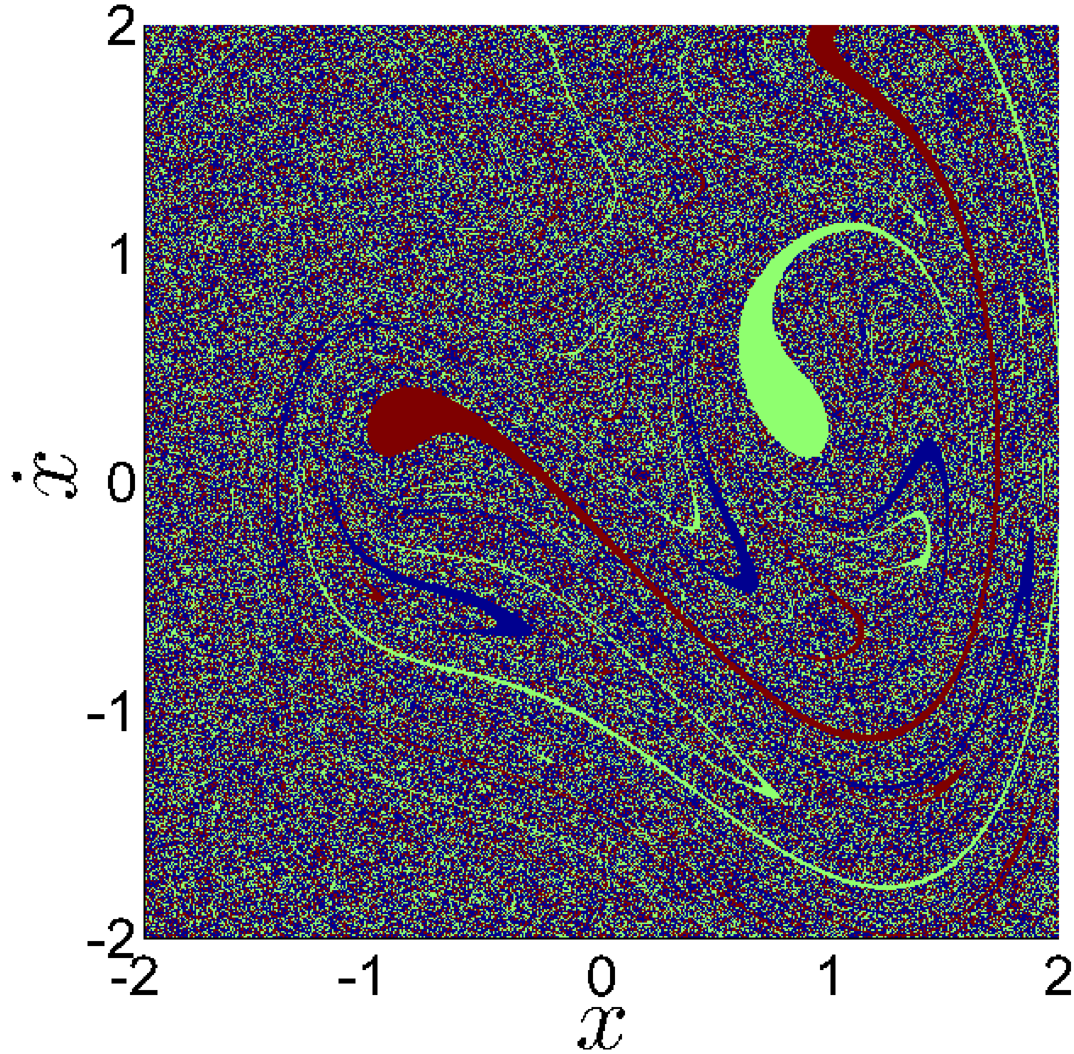

Figure 5.2. The basins of attraction of the Duffing oscillator of Eq. 1. A fine grid of 2000×2000 of initial points is considered and different colors are chosen according to which attractor an initial condition goes to. The points that end up in the $P3$ attractor are colored in blue. The points that go to the P1R attractor are colored in green. The initial conditions whose final state is $P1L$ are colored in red.

our case it is very difficult to use this analytical approach since we do not know the explicit expression of the time-2π map of the Duffing oscillator. For that reason, in the following sections we have used a much more quantitative procedure to compute the probabilities of the final state of the system. We have done this by directly sampling the entire phase space with a uniform grid of initial conditions, and computing the ratio of the number of points (of those initial conditions) ending in a particular final state relative to the number of the sample. Interestingly, we have found that this method works even for very low resolution samples.

We have divided the statistical analysis of the model into two parts. First, we have studied the probabilities obtained by sampling the phase space along horizontal (or vertical) one-dimensional straight lines. Second, we have used a two-dimensional grid covering the whole phase space to compute the probabilities associated with each attractor in phase space.

5.3 One-dimensional analysis

Our goal in the first analysis is to compute the probabilities of ending on a given attractor, assuming that we know only one of the two coordinates of the initial condition, either x or $\dot{x}$. This means that we need to compute a conditional probability. We could do this formally by taking as our initial probability density function ρ_0 the one that is zero everywhere except in the coordinate that we know, and recursively applying the Perron-Frobenius operator until it converges. We then integrate the resulting probability density function in the neighborhood of each attractor to find the conditional probabilities

$$P(P1L|x = x_i), \quad P(P1R|x = x_i), \quad P(P3|x = x_i). \tag{5.4}$$

These are the conditional probabilities of ending up in each attractor given that we know the coordinate x_i of the initial condition. However, as we have already mentioned, finding the final probability density function using the Perron-Frobenius operator is usually very complicated.

An alternative way to compute these conditional probabilities is to sample the phase space in x_i along $\dot{x}$ with a uniform one-dimensional grid, and then compute the final state of all of those initial conditions. Then taking the ratio of initial conditions that belongs to each basin of attraction gives us a really good aproximation of $P(P1L|x = x_i)$, $P(P1R|x = x_i)$ and $P(P3|x = x_i)$. We have done this for every x_i in different resolutions. We have followed a similar procedure to compute the conditional probabilities $P(P1L|\dot{x} = \dot{x}_i)$, $P(P1R|\dot{x} = \dot{x}_i)$ and $P(P3|\dot{x} = \dot{x}_i)$, where we assume that we know the coordinate $\dot{x}_i$ of the initial condition. We summarize the results in the diagrams and graphs shown in Figs. 5.3.

As we can see in Figs. 5.3, for the resolutions 300×300 and higher, the conditional probabilities remain constant for almost every x_i. It is also clear in Figs. 5.3 that the period-3 attractor (blue basin) is the most probable attractor and the two period-1 attractors have almost the same probability (around 0.25). For the interval $\dot{x} = [0, 0.5]$, a big change in the trend occurs, when $P(P3|\dot{x} = \dot{x}_i)$ (blue) loses over 24% of its value and $P(P1L|\dot{x} = \dot{x}_i)$ (red) sums 33% to its value. In this interval, the $P1L$ attractor (red) is the most common, additionally $P(P1R|\dot{x} = \dot{x}_i)$ (green) sums 20% too and becomes more frequent inside this interval. The location of the two large basin cells of the period-1 attractors in phase space lies inside this interval, which explains the new trend of probabilities. In the interval $x = [0.8, 1]$ another big change in the trend occurs when $P(P3|x = x_i)$ (blue) loses about 25% of its value and $P(P1R|x = x_i)$ (green) attractor increases its value by 55% of its value and

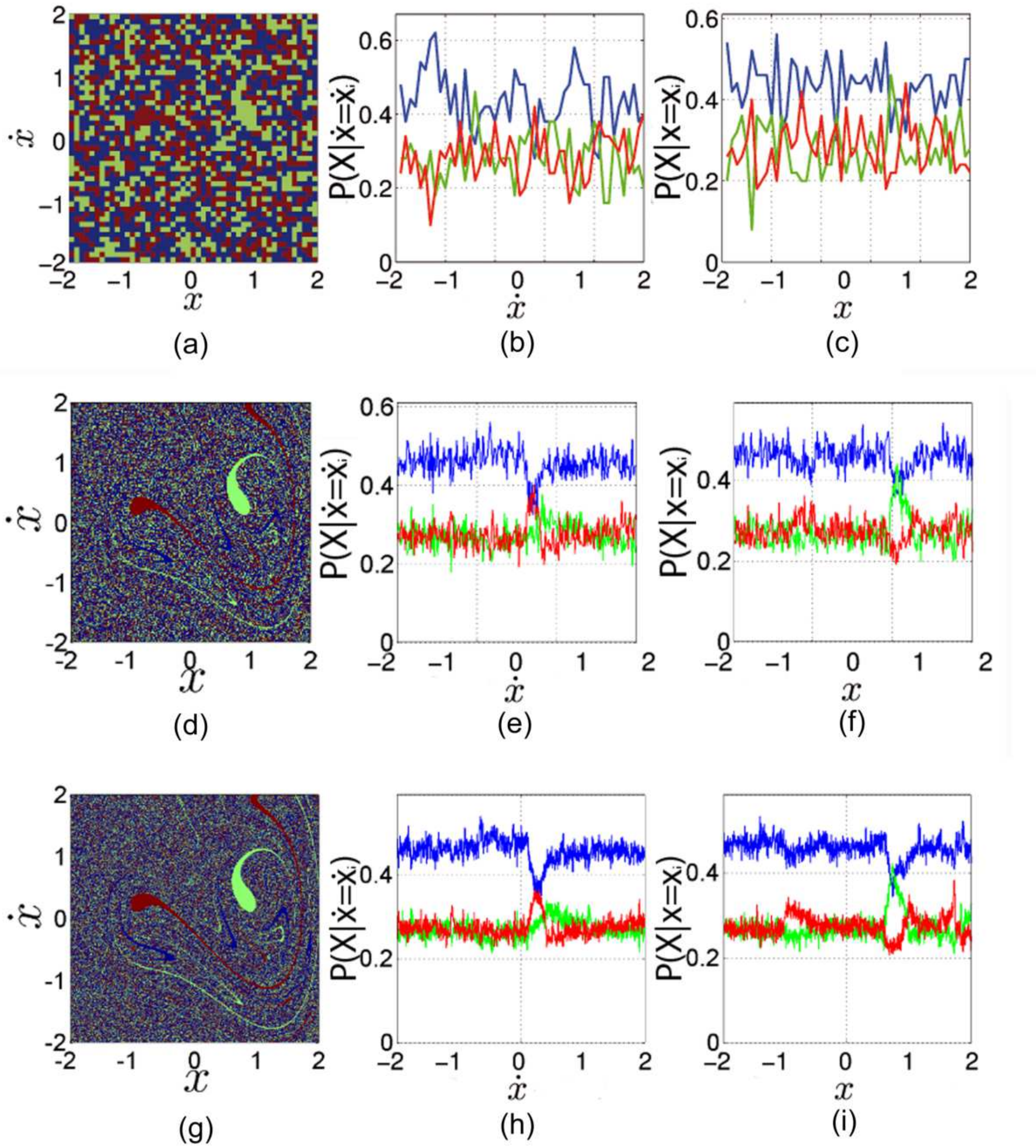

Figure 5.3. Conditional probabilities of each attractor in the Duffing oscillator for different resolution grids. Panels (a), (d) and (g) are the basins of attraction of the Duffing oscillator for grids of 50×50, 300×300 and 1000×1000 correspondingly. The points are colored according to which attractor an initial condition goes to. In panels (b), (e) and (h) we have plotted the conditional probabilities associated with each vertical line of the phase space, $P(P1L|\dot{x} = \dot{x}_i)$ (red), $P(P1R|\dot{x} = \dot{x}_i)$ (green) and $P(P3|\dot{x} = \dot{x}_i)$ (blue). In panels (c), (f) and (i) we have plotted the conditional probabilities associated with each horizontal line of the phase space, $P(P1L|x = x_i)$ (red), $P(P1R|x = x_i)$ (green) and $P(P3|x = x_i)$ (blue).

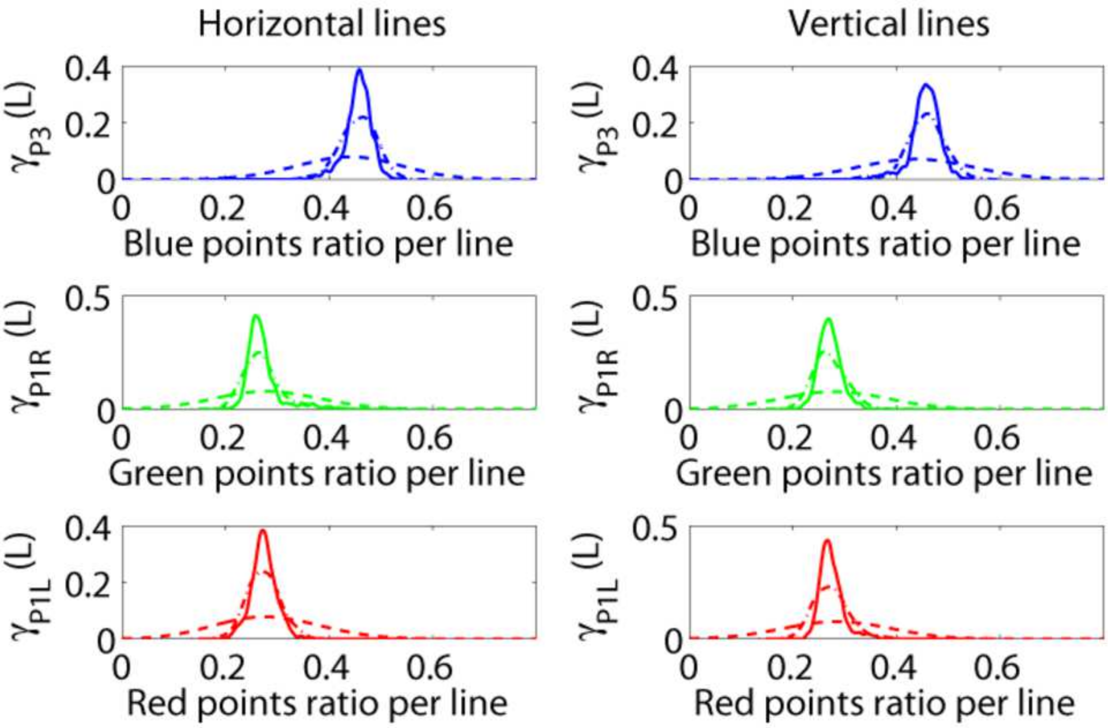

Figure 5.4. Kernel probability density estimation (KPDE) of the points ratio associated with each attractor for a horizontal or a vertical line. This figure shows the KPDE related to each attractor for a random picked horizontal or vertical line. The horizontal axis measures the ratio points per line. The vertical axis represents the probability density of each attractor, for horizontal lines on the left and for vertical lines on the right. Here, γ_{P3} (blue) is the probability density function associated with the $P3$ attractor, γ_{P1R} (green) is the probability density function associated with the $P1R$ attractor and γ_{P1L} (red) is the probability density function associated with the $P1L$ attractor. We have repeated the same computation for the following resolutions: 50×50 as dash line, 300×300 as dot dash line and 1000×1000 as solid line.

becomes the most frequent attractor in phase space. Additionally, in the interval $x = [1.3, 1.6]$ there is a peak in $P(P1L|x = x_i)$ (red). This attractor sums over 40% of its value and becomes the most common attractor in this small interval. Again this result arises from the location of the basin cells of the period-1 attractor in phase space. However, despite of those big local changes of the conditional probabilities near to the large basin cells, we find that in the rows or columns with a strong Wada property (which are the most common) the conditional probabilities are almost constant. The conditional probabilities only change abruptly in the regions with big basin cells.

We can also treat the total length (found in a given horizontal or vertical straight line in the phase space) associated to the basin with each attractor, as a continuous random variable which will have associated a probability density function (pdf), $\gamma(L)$. These pdfs allow us to compute the probability that the length of each attractor of a horizontal (or vertical) straight line (in phase space) is within a $\Delta\delta$ interval, this is $P(L-\Delta\delta < L < L+\Delta\delta)$. We have obtained the pdfs by counting the number of initial conditions for every one-dimensional horizontal (or vertical) straight line that goes to each attractor, and computing later the histogram of the number of horizontal (or vertical) lines vs the number of initial conditions. Normalizing this histogram by the number of horizontal (or vertical) lines we obtain the desired pdf. We can show the results of the computed pdfs for different resolutions in Fig. 5.4. The length associated with each attractor in every straight line is measured by the number of initial conditions. In order to compare the different resolutions, we have also normalized the horizontal axes where we represent the ratio of points per line. As we can see, as the phase space resolution increases the pdf shapes become smoother. On the one hand, from the statistical coefficients calculated from the data, we can conclude that the pdf associated with the length of the $P3$ attractor in either vertical or horizontal straight lines is not normally distributed and has a long tail in the left side. On the other hand the pdfs associated with the lengths in horizontal (or vertical) straight lines for the $P1L$ and $P1R$ attractors, are not normally distributed either, and have a long tail on the right side. As expected, the mean of the pdf associated with the $P3$ attractor doubles the mean of the $P1L$ and $P1R$ attractors, either in the horizontal or the vertical direction. Interestingly, the standard deviation is about the same for all the pdfs - in both the horizontal and vertical directions.

5.4 Two-dimensional analysis

In the two-dimensional case the invariant probability density function would be computed taking as our initial probability density function ρ_0, the one that is one everywhere in the square $[-2,2][-2,2]$, and applying recursively the Perron-Frobenius operator until it converges. We would integrate the resulting probability density function in the neighborhood of each attractor to find the total probabilities,

$$P(P1L), \qquad P(P1R), \qquad P(P3). \tag{5.5}$$

These are the total probabilities of ending in each attractor assuming that we do not know any of the coordinates of the initial conditions. However, as in the previous case, to find the final probability density function using the Perron-Frobenius operator is usually difficult.

An easy way to approximate the total probabilities is taking a uniform two-dimensional grid and computing the ratio of initial conditions that belongs to each basin of attraction. We have done this for different resolutions of the grid as we can see in Fig. 5.5, where it can be clearly observed that the pattern of the basins of attraction is almost stable for resolutions higher than 300×300. All the basins of attraction keep their shape near the location of the attractors, but as we move away from them, they begin to mix and become fractal. In Table I we summarize the number of initial conditions taken for every resolution and going to each attractor.

Table 5.1. Points per basin.

Phase space size	Precision	Initial points	P3 (Blue)	P1D (Green)	P1I (Red)
50×50	0.08	2,500	1,092	700	708
100×100	0.04	10,000	4,623	2,687	2,690
200×200	0.02	40,000	18,248	10,845	10,907
300×300	0.0133	90,000	41,089	24,326	24,585
400×400	0.01	160,000	78,158	43,218	43,623
500×500	0.008	250,000	114,298	67,780	67,922
$1,000 \times 1,000$	0.004	1,000,000	455,877	270,612	273,511
$2,000 \times 2,000$	0.002	4,000,000	1,826,335	1,081,648	1,092,017
$3,000 \times 3,000$	0.00133	9,000,000	4,104,916	2,434,470	2,460,614

For very low grid resolutions there is a large change in the probabilities going to each attractor. But beyond a given threshold in the resolution, the probabilities remain constant. This is what we show in Fig. 5.6. We can clearly see how, for a resolution of 300×300 or higher, the probabilities converge to constant values. The total probability of landing in the period-3 attractor $P(P3)$ (blue basin) converges to 0.456 (45.6%), the total probability of landing in the period-1 attractor to the right $P(P1R)$ (green basin) converges to 0.270 (27%) and the total probability of ending in the period-1 attractor to the left $P(P1L)$ (red basin) converges to 0.274 (27.4%). This clearly indicates that the results are robust and can be used in the statistical prediction that we are looking for. As expected, due to the convergence of the Perron-Frobenius operator these probabilities are scale free. In Fig. 5.7 we show how the probability of each attractor changes depending on its location over the phase space. We have divided the highest resolution phase space into $90,000$ samples squares where each square possesses a sample of 100 grid points. Then, we have computed the relative frequency of each attractor in each square and we have used a heatmap to represent the associated probabilities where brighter red color means high probability and darker red color means less probability. Now we can actually visualize why and even where the probability of being in the basin of the period-3 attractor, for example, is greatest over the phase space. The orange color on the left panel in Fig. 5.7 illustrates how the high probability of the period-3 attractor dominates in the fractalized zones, while in the other two panels the dark red color illustrates the low probability of the period-1 attractors over the same palaces in the phase space. The fact that the fractal zones occupy a larger area of the phase space explains why at the aggregate level we obtain the results above. We can state that the long term dynamics of this system depends on the attractor that governs in the fractal zones.

Surprisingly, we find here a very remarkable result in the rows and columns with

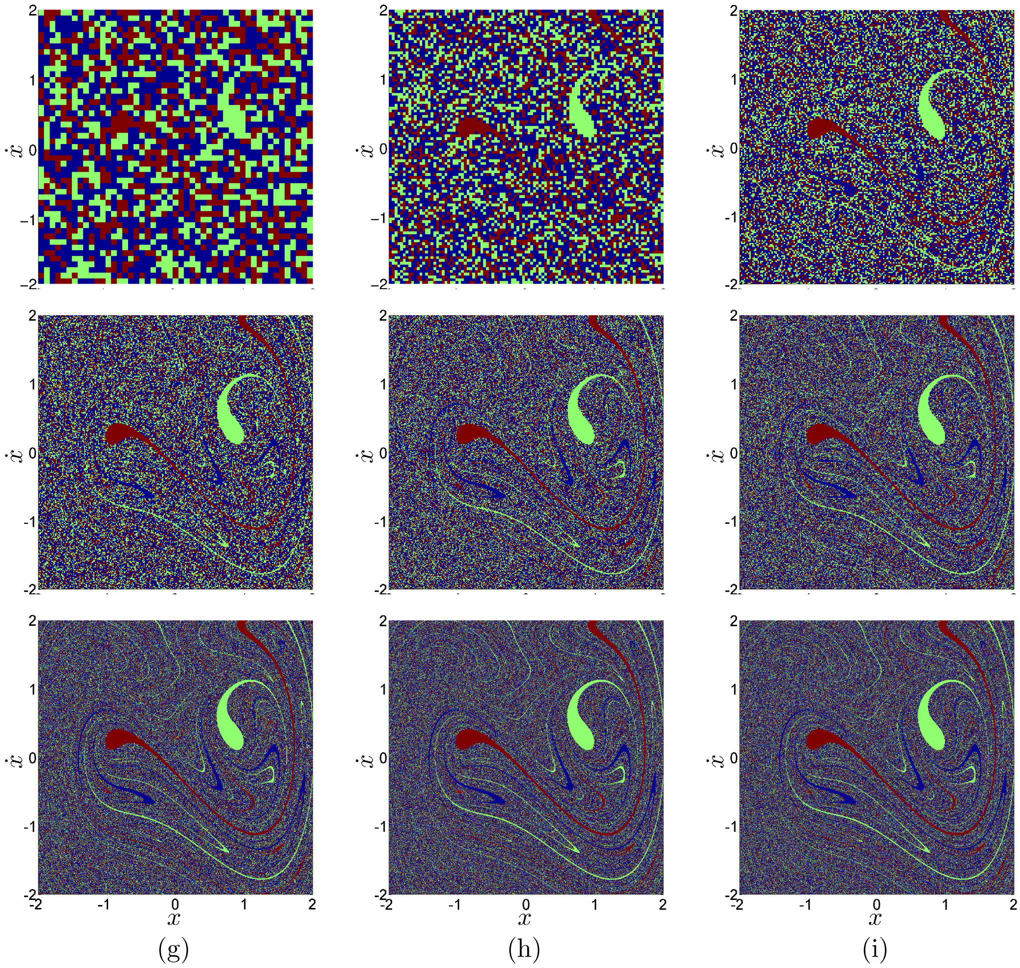

Figure 5.5. Basins of attraction for different resolutions. The picture shows the basin of attraction of the Duffing oscilator for different resolutions, (a) 50×50, (b) 100×100, (c) 200×200, (d) 300×300, (e) 400×400, (f) 500×500, (g) 1000×1000, (h) 2000×2000 and (i) 3000×3000.

a strong Wada property. In those regions it is satisfied that

$$\begin{aligned} P(P1L|x = x_i) \approx P(P1L|\dot{x} \approx \dot{x}_i) \approx P(P1L) \\ P(P1R|x = x_i) \approx P(P1R|\dot{x} = \dot{x}_i) \approx P(P1R) \\ P(P3|x = x_i) \approx P(P3|\dot{x} = \dot{x}_i) \approx P(P3). \end{aligned} \tag{5.6}$$

This means that in the regions with the Wada property the knowledge of one of the coordinates of the initial condition does not improve our prediction capability. It is the same as not knowing any of the coordinates of the initial condition. The conditional probability and the total probability differ only in the regions with large basin cells.

As we have just seen, the probabilities of each basin of attraction converge to

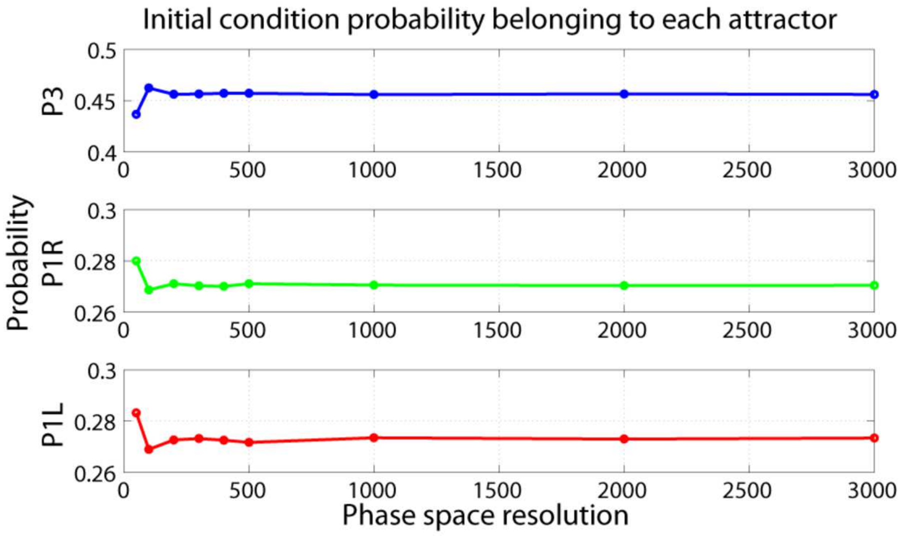

Figure 5.6. The attractors probability trends. This figure shows the probabilities of each basin of attraction in the phase space (vertical axis) and the resolutions of each phase space (horizontal axis). The blue line represents the initial condition probability belonging to the period-3 attractor $P3$, the green line corresponds to the period-1 attractor $P1R$, and the red line to the period-1 attractor $P1L$.

a constant probability when the resolution of the phase space increases as shown in Fig. 5.6. It seems that improving the resolution does not affect the probabilities anymore. To show that this result does not have any grid dependence, we have carried out a Monte Carlo simulation. We have chosen the Monte Carlo method because the error on the results typically decreases as $1/\sqrt{N}$ [114].

To implement the Monte Carlo method, we have chosen randomly $50,000$ initial conditions with 15 decimals precision in phase space. This precision is equivalent to a fine grid of $10^{16} \times 10^{16}$ initial points. We have used a uniform probability distribution to generate the initial conditions as shown in Fig. 5.8. Then, we have integrated them using a fourth-order Runge-Kutta integrator with a fix integration step of $2\pi/200$ and classifying them depending on the attractor towards which they converge. We have chosen bigger integration steps for this calculation because of the high precision of initial conditions. Next, we have computed the ratio of initial conditions going to each attractor versus the total number of initial conditions in the sample to obtain the attractors probabilities. We have obtained the following results; the total probabilities of the period-1 attractors $P1R$ and $P1L$ are 0.270 and 0.272 respectively. The total probability of the period-3 attractor is 0.458. These probabilities are almost identical to the probabilities found in the statistical analysis found with the uniform grid. With this result, we can confirm that the probabilities obtained with the uniform grid for the statistical predictions of an arbitrary initial condition are very accurate.

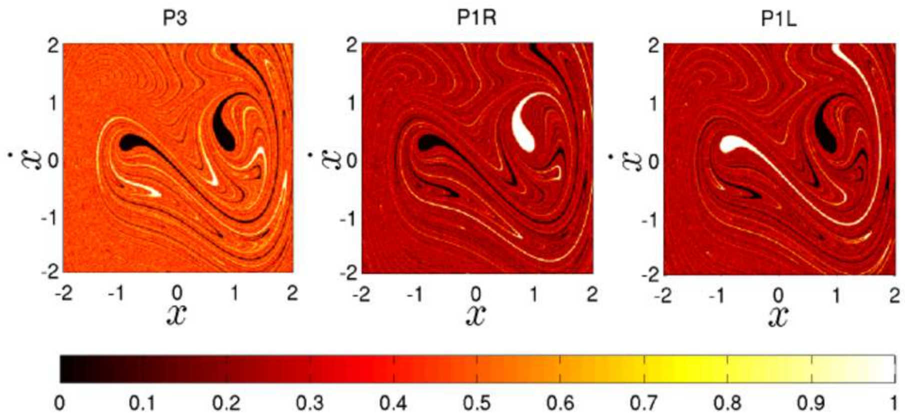

Figure 5.7. Fluctuation of the local probabilities of the attractors on the phase space. We have divided the original phase space into 90, 000 samples squares where each square possesses a sample of 100 grid points from the the original 3000×3000 phase space. Then, we have computed the relative frequency of each attractor in each square over the phase space. We have plotted the probability of each attractor on the phase space separately. We have plotted $P3$ on the left panel, $P1R$ on the central panel and $P1L$ on the right panel. When there is 100 percent probability for the square to fall into a particular attractor we have colored the square in white. When there is no chance (probability 0) for a particular square to fall into a particular attractor we have colored the square in black. When the probability of falling into a particular attractor is between 0 and 1 we have colored the square in a red scale color, where dark red is close to probability 0 and yellow is close to probability 1. Note that here we are not plotting the basins of attraction, but the spatial probability function associated with each attractor.

5.5 Fractal boundaries and probabilities

There is a very intuitive explanation for the convergence of the total probability of each attractor towards a constant value, as shown in the previous sections. The fractal basins that we study here have a fractal dimension, though we have not computed it since it is not relevant for the statistical predictions that we were studying. Simply by looking at Fig. 5.5 it is clear that we face self similar basins of attraction that do not change with the scale at which they are measured [115]. The method used to compute the probability of each basin of attraction in phase space, is somehow like measuring the area that each basin occupies in the phase space. This is similar to what happens in the famous coastline paradox [36]. As we increase the resolution, the perimeter of the coastline increases towards infinity. But the area enclosed by that perimeter remains constant [116, 37]. A completely analogous

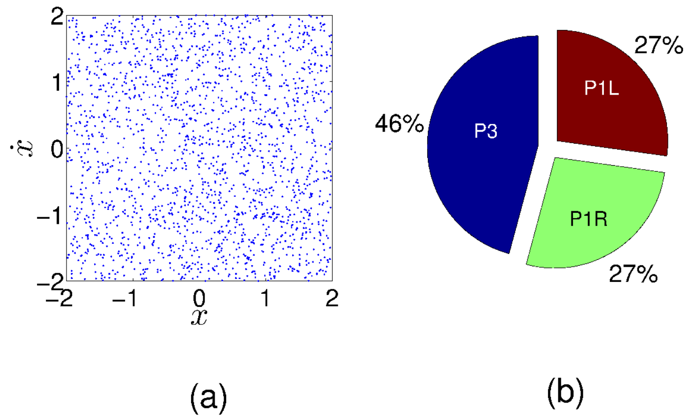

Figure 5.8. The probability of the attractors when the initial conditions are chosen randomly. (a) Shows the initial conditions taken in the Monte Carlo method to compute the probability associated to each attractor. The pie diagram in (b) shows the probability of each attractor in the phase space according to this sample of initial conditions.

behavior is found in the case of the Duffing oscillator. We get more points on the basin boundaries and the precision of each point increases as well. But the area of each basin of attraction occupies the same space in all scales of the phase space from a given threshold resolution. This behavior is helpful when we are interested in the global dynamics of the system. In some dynamical systems with sensitive dependence on initial conditions, knowing the attractor's probabilities is enough to understand the system and to do statistical predictions. In many cases, making clever decisions in accordance to the probability of every attractor in the system is good enough.

5.6 Conclusion

In this chapter, we have studied the Duffing oscillator model with a choice of parameters showing the Wada property. Then, by using methods from statistical analysis, the probabilities of ending up in a particular attractor of the phase space have been found. We have also shown that these probabilities might be scale invariant. This result is related to the fractal nature of the basins boundaries. A Monte Carlo simulation has been used to verify the values of the attractor probabilities and we have found that are very similar to the values calculated in the statistical analysis. We have shown that knowing the attractors probabilities in some cases is enough to predict the future state of the system and to tackle the final state sensitivity problem, even if we do not have any knowledge about the initial conditions of the system. We have also shown how relatively low grid resolutions (300×300 or higher) are enough to obtain the statistical information needed for the statistical predictions about the future state of the system, even in systems as complicated as the Duffing oscillator with the Wada property. This means that we can save a lot of time, effort and memory space when computing the probabilities associated with this kind of systems. Finally, we have also seen how in terms of prediction, the knowledge of one of the coordinates of the initial condition, provides similar results to the case when the two coordinates in the Wada regions are unknown. The technique presented here can be applied to any dynamical system with fractal basins or even Wada basins over its phase space. We believe that using this technique with relative low resolution phase space samples might give a good understanding of the attractors distributions and probabilities helping decision makers and researchers to make decisions and even predict, optimizing their computational time and resources.

Chapter 6

Results and Discussion

"And now someone had uncoiled the angle of that destiny, had set it on another path, had given the ants the secret of the wheel, the secret of working metals-how many other cultural handicaps bad been lifted from this ant hill, breaking the bottleneck of progress?"
-Clifford D. Simak, *City*

Economic systems have changed dramatically in the past century. Probably, the technological progress is one of the main drivers for this change. Scientists understood the power of computational technology a long time ago, and they have started to use it almost immediately, computing very complicated mathematical equations or making different virtual models of reality. Computational models give insights into any system that can be written in code. But when the system in question, is a complex system, this tool is one of the most useful tools to study the dynamics that emerge in such systems. Markets or economic organizations by definition are complex systems and in this thesis, we have used some of the most advanced techniques to model and to study the intrinsic complexities of these systems.

Inspired by the complex behaviours of some supplying firms, we have developed the *supply based on demand dynamical model*. This simple model studies how firms behave under uncertainty of demand forecast. It is an iterative model where the firms must produce all their stock before they know how many products their consumers will consume. We have found that the model is capable of reproducing many types of dynamics such as equilibrium, limit cycles, chaos, and even transient chaos under simple and reasonable economic assumptions. We have found a positive Lyapunov exponent in a wide range of values of the elasticity of demand and the margin set by the firm. Additionally, small variations in these parameters will lead to very different market dynamics. We have shown that the transient chaotic dynamics that arise naturally in the model are a good approximation of a market crash. We assume that the elasticity of demand is given by the preferences and the utility functions of the consumers, but the margin represented by the parameter M is set by the firm, and this peculiarity allows the firm to adjust the price in light of the market circumstances. Applying the partial control method on the price we have shown that the firm can control the market, preventing the collapse, even when external disturbances are present in the system. Additionally, the firm can apply the partial control strategy on the quantity supplied in some special cases and while it is doing that, it is able to extend the natural barriers of the system supplying more goods than before. The partial control method has been found suitable for applying on the demand as well, when a firm with high market power intervenes directly in the market, buying the excess supply, or advertise to encourage the consumption in the market. In all cases we have sustained the dynamics in the transient regime with a term of control much smaller than the external disturbances introduced in the model.

In our daily life we face complex decision problems, characterized by being highly dependent on other's decisions or to some uncertain payoff. The *El Farol Bar problem* introduced by the Irish economist Brian Arthur was one of the first attempts to explore these situations. We have studied a model in which 50 agents must locate themselves in a pool that pay a payoff that depends on the attendance to the pool and on some stochastic function. To analyze this model we have developed an agent based model that takes into account 13 different strategies. After grouping all these strategies in four different groups, depending on the conceptual approach, we have found that the naïve behavior is the most successful, in particular the *Always low pool* strategy. This strategy has been found to be the most profitable strategy, even when the cost of switching between the pools is very high or when we increase the amount of agents to 73. We have shown that when we increase the diversity of strategies in the system, the attendance converges very fast to the optimal attendance of each pool. When only 13 different strategies are present in the system the attendance converges in only 10 iterations. This is an interesting result since we have assumed that more strategies are needed to observe this dynamics.

Although this technique of modeling uncertainty is very beneficial in some complex social situations, uncertainty can be very difficult to define even in deterministic systems. We have studied the Duffing oscillator model with a choice of parameters showing the Wada property. We were interested in the statistical properties of the system when it poses the Wada basins. We have computed the probabilities of ending up in a particular attractor of the phase space, showing that these probabilities might be scale invariant. This means that globally we can succesfully predict the system, even if we do not have any knowledge about the initial conditions of the system. We have also shown how relatively low grid resolutions (300×300 or higher) are enough to obtain the statistical information needed for the statistical predictions in the Duffing oscillator with the Wada property. We have also shown that in the Wada basins, knowing half of the information about the initial condition (one coordinate) does not help us to make predictions about the future state of the system.

6.1 Future work

In this thesis, we have explored the dynamics produced by the SBOD model when one firm interacts with one consumer. An interesting question is what kind of dynamics will appear in an aggregated market whith many consumers and firms that follows the same rules of the SBOD model. We think that the right approach to this problem is using an agent based model because it offers a lot more flexibility in modelling the consumers and firm's behaviors. Another idea that deserves more exploration is if the partial control method would be suitable in such aggregated market.

Chapter 7
Conclusions

> *"The noblest pleasure is the joy of understanding."*
> -Leonardo da Vinci

Here we summarize the main results of the present thesis:

1. Economics has changed considerably in the last century. The main force that drove this huge change is without doubt, the computational revolution. Advances in hardware and software have changed forever the way humans interact, transfer value or communicate. Besides changing the reality itself, this revolution has changed the way in which scientists observe and study the economy, letting them develop new paradigms and models that fit a lot better in some of the most complex economic phenomena.

2. We have shown that the *supply based on demand dynamical model* is capable of producing a rich variety of dynamics; fixed points, limit cycles, chaos and transient chaos, under simple and reasonable economic assumptions. We have shown that the price elasticity of demand and the margin imposed by the firm have huge effects on the market dynamics. Small variations in these two parameters will lead to different market behaviors. Finally, we have shown that the final bifurcation can be a good approximation to market collapse.

3. We have applied the partial control method to the *supply based on demand dynamical model* in the presence of disturbances, for a particular choice of parameters where it shows transient chaos. Typical uncontrolled trajectories in this system follow a transient chaotic motion until they escape to an external attractor. In the context of our model this escaping dynamics mean a market collapse. With the goal of preventing the market collapse, we have applied the partial control method in three different ways. We have begun applying the

partial control on the price imposed by the firm, then we have applied the control on the quantity demanded and supplied. We have obtained the respective safe sets with the Sculpting Algorithm and successfully prevented the collapse in all cases where the control was a lot smaller than the disturbance.

4. We have studied the system proposed by the Complexity Challenge Team, which can be considered as a more complex *El Farol bar problem*. We have developed an ABM using Netlogo software in which we code 13 different strategies that we grouped in four families. We have found that the most successful families are the naïve and the pattern based families. This means that in this system when the agents use many different strategies, decisions that are built based on patterns in the data or repetition lead to a higher balance on average. We have found that the most poromising strategy was the *Always low pool* even when τ increases. The *Always stable pool* strategy becomes the best when N increases over 73 agents. We have also shown that in spite of the relative small number of strategies (13 strategies), the average N in each pool approaches the attractor of $N = 20$ in the low and in the high pool, following the same dynamics predicted by Brian Arthur in his original paper on *El Farol bar problem*.

5. We have studied the Duffing oscillator model for a choice of parameters showing the Wada property. We have shown that knowing the attractors probabilities in some cases is enough to predict the future state of the system and to tackle the final state sensitivity problem, even if we do not have any knowledge about the initial conditions of the system. We have demonstrated how relatively low grid resolutions 300×300 or higher, are enough to obtain the statistical information needed for the statistical predictions about the future state of the system, even in systems as complicated as the Duffing oscillator with the Wada property. Finally, we have shown how in terms of prediction, the knowledge of one of the coordinates of the initial condition, provides similar results to the case when the two coordinates in the Wada regions are unknown. The technique presented here can be applied to any dynamical system with fractal basins or even Wada basins over its phase space.

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Resumen de la tesis en castellano

Introducción

En los últimos años hemos sido testigos de un amplio abanico de avances tecnológicos que han cambiando nuestra forma de entender la economía. Entre estas tecnologías quiero destacar los libros contables distribuidos - en particular el Blockchain de Bitcoin y los contratos inteligentes de Ethereum, el aprendizaje automático y el análisis masivo de datos, las redes sociales y los modelos empresariales de economía compartida. Todos estos avances tienen tres cosas en común, Internet, coste marginal cero de la información digital y por último sistemas y redes de computación muy avanzados. Estas jóvenes tecnologías representan y a la vez son el fruto de un cambio de paradigma en las ciencias económicas. De la teoría económica centrada en la producción que ha sido la base intelectual y práctica que sostuvo el crecimiento económico exponencial vivido en los últimos tres siglos, estamos pasando rápidamente a otro tipo de economía que se puede dividir en tres clases, la economía de la información, la economía de la atención y la ingeniería económica de sistemas descentralizados.

Para estudiar y analizar estos fenómenos económicos debemos ampliar nuestra caja de herramientas y reciclar nuestras teorías. Por esta razón muchos economistas en las últimas décadas comenzaron a estudiar la economía bajo otro prisma, el prisma de la complejidad. La ciencia de la complejidad comenzó a desarrollarse en el siglo pasado y fue evolucionando junto con la computación. Naturalmente las herramientas que ofrece esta nueva ciencia, como por ejemplo simulaciones computacionales de diferentes tipos, parecen más adecuadas para estudiar fenómenos económicos antiguos y también los emergentes. En este trabajo estudiamos la economía desde el punto de vista de la complejidad, utilizando otro paradigma, modelos no lineales que describen la economía como un sistema abierto compuesto por unos agentes heterogéneos con racionalidad limitada, tomando decisiones en contextos muy específicos que dan lugar a redes de interacciones y fenómenos emergentes. Esta economía está sumergida en una dinámica compleja y fuera de equilibrio evolucionando y cambiando constantemente. Esta tesis tiene como objetivo estudiar las complejidades de la economía utilizando diferentes técnicas de simulación computacional.

Metodología

Las herramientas utilizadas para elaborar este trabajo de investigación han sido fundamentalmente de carácter teórico-computacional. Se han utilizado y desarrollado diversos modelos matemáticos no lineales. La formulación matemática de estos modelos ha sido en forma de mapas discretos y ecuaciones diferenciales ordinarias. Para su resolución se han empleado métodos tipo Runge-Kutta en la mayoría de las ocasiones, ajustando las características del integrador a cada caso, de tal modo que se obtuviera el mejor compromiso posible entre fiabilidad de los resultados y tiempo de cómputo. Además, para realizar los cálculos más pesados se han utilizado los recursos del Grupo de Dinámica No Lineal, Teoría del Caos y Sistemas Complejos de la URJC, servidores de alto rendimiento y un cluster que permite paralelizar los procesos obteniendo un poder de computación superior. Los modelos basados en agentes se desarrollaron utilizando el software Netlogo y el análisis de datos posterior se ha realizado con el software R. La programación se ha realizado fundamentalmente en Matlab, R, Netlogo, C, C++, Fortran y MPI para los procesos en paralelo.

El método de suministro en función de la demanda

El modelo estándar de la demanda y la oferta es de naturaleza estática y asume que la demanda y la oferta son dos observables independientes uno del otro. La única forma de influir en la forma o la posición de las curvas de la demanda y la oferta es a través de una fuerza exterior al propio sistema. Muchas veces se utiliza la tecnología como un fenómeno exógeno al mercado capaz de mover la curva de la oferta de su posición original. Pero en la realidad uno de los factores más importantes que determinan la oferta es la propia demanda o mejor dicho la demanda estimada por los vendedores. Hoy en día muchas firmas de todos los tamaños invierten muchos recursos en el estudio de la demanda porque esta información es crucial para el buen funcionamiento de la empresa. El famoso problema que se centra en producir de acuerdo con la demanda estimada se conoce en la literatura como el *problema del vendedor de periódicos* y se remonta a finales del siglo XIX. Desde entonces y hasta hoy se han publicado muchísimos artículos que analizan matemáticamente este problema, pero muy pocos que investigan el efecto de este comportamiento de las firmas en la dinámica del mercado. Desarrollomos un modelo dinámico no lineal de la oferta basada en la demanda y exploramos las dinámicas que este produce. A continuación, presentaremos una lista de resultados y conclusiones de esta investigación:

1. Desarrollamos el modelo dinámico de la oferta basada en la demanda. Este modelo simula un mercado en donde los compradores obedecen la ley de la demanda, es decir cuando más barato es el producto en venta, más cantidad quieren consumir y los vendedores en cada iteración intentan predecir la de-

manda futura para producir los productos exactos que se vayan a consumir. De esta manera la firma consigue maximizar sus beneficios. Para estudiar la dinámica que produce este modelo desarrollamos dos comportamientos de proveedor muy básicos: el proveedor ingenuo y el proveedor cauto y optimista. Hemos modelado estos comportamientos matemáticamente utilizando la señal de éxito en la iteración anterior y la función no lineal de producción del proveedor. Demostramos que, aunque sean comportamientos muy sencillos, cuando se iteran producen una dinámica de mercado muy interesante.

2. Describimos una función cuadrática convexa de producción del productor. Basándonos en la observación de que en algunos casos el coste de producción de un producto baja con la cantidad producida hasta un punto en donde el productor necesita más capital para seguir produciendo y esto encarece el coste de producción de los productos subsecuentes.

3. Mediante simulaciones numéricas, demostramos que para algunos valores de los parámetros se producen dinámicas de mercado caóticas que en algunos casos son transitorias. Por ejemplo, cuando la firma aumenta el margen de beneficios a un nivel determinado la demanda empieza a fluctuar caóticamente y esto rápidamente se traduce en trayectorias caóticas del precio en el mercado. Confirmamos estas observaciones calculando los exponentes de Lyapunov y los diagramas de bifurcación del sistema en estos valores de parámetros.

4. Analizamos la influencia de la elasticidad de la demanda en la dinámica del mercado. Demostramos que este parámetro se caracteriza por tener un efecto muy importante sobre la dinámica del mercado, pequeñas variaciones en este parámetro producen dinámicas muy diferentes.

5. La aparición de caos transitorio en este sistema nos llevó a extender nuestras primeras suposiciones del modelo, prohibiéndole a la cantidad ofertada o demandada obtener valores negativos. De esta manera demostramos que la bifurcación final significa un colapso de mercado.

Evitando el colapso utilizando el control parcial

Los colapsos económicos son unos de los eventos más catastróficos en las sociedades modernas. Las heridas que nos ha dejado la crisis financiera del 2008 no han cicatrizado aún y nos sirven para recordar que nuestros sistemas económicos son muy vulnerables a eventos trágicos de esta naturaleza. Existe una extensa literatura alrededor de este fenómeno económico y se centra en tres líneas principales de investigación. La primera analiza las diferentes causas que encadenan una crisis económica, la segunda se centra en la predicción de este tipo de evetos, y la última investiga cómo aplicar control sobre el sistema económico para prevenir colapsos de este tipo en el futuro. En este capítulo seguimos la tercera línea de investigación y aplicamos con éxito por primera vez en la economía un método de control llamado *control parcial*. Esta es de las pocas evidencias de que es posible controlar el mercado de abajo arriba, sin la necesidad de un órgano central. Estos puntos resumen lo esencial de esta investigación:

1. Describimos brevemente el modelo dinámico de la oferta basado en la demanda con énfasis en la existencia de caos transitorio para algunos valores de los parámetros y explicamos porque la última bifurcación significa colapso del mercado.

2. El método utilizado en este capítulo para controlar este modelo se basa en un algoritmo llamado *el algoritmo del escultor*, que permite encontrar conjuntos seguros, en los cuales el control aplicado sobre el sistema es menor que el ruido. El ruido añadido al sistema es un término estocástico que sigue una distribución uniforme y simula la incertidumbre que podemos encontrar en sistemas económicos. Explicamos con detalle el funcionamiento del método y como se aplica a nuestro modelo.

3. Controlamos con éxito las trayectorias del precio en el caso del productor ingenuo. Una de las principales suposiciones del modelo SBOD, es que el productor es el único capaz de fijar y cambiar los precios de los productos. El productor actúa de *price maker* y el consumidor es *price taker*. En este contexto el productor puede variar el precio del producto en cada iteración, subiéndolo, aumentando el margen de beneficio o bajándolo y reduciendo su margen de beneficio. De esta manera también con la presencia de ruido en el precio, el productor ingenuo logra evitar una explosión de precio manteniendo el precio en la región transitoria y así evitar el colapso del mercado. Este tipo de control es un control de abajo arriba, es decir, ningún órgano central aplica el control desde arriba sobre el mercado, sino la misma firma lo hace.

4. Aplicamos con éxito el control parcial sobre la demanda en el caso del productor ingenuo. La demanda es mucho más difícil de controlar porque depende de las preferencias individuales de los consumidores. En este contexto aplicamos el control parcial directamente sobre la demanda como una intervención directa de la empresa en el mercado. Es decir, si hay poca demanda la empresa por un lado puede adquirir la cantidad necesaria de productos que prevengan que la demanda caiga a cero. Por otro lado, la empresa puede influenciar la demanda invirtiendo en publicidad que incentiva o desincentiva la demanda, de acuerdo con la demanda necesaria. Demostramos también que en casos en donde existe una firma que tiene un proveedor principal este método de control también puede ser muy útil.

5. Aplicamos el control parcial sobre la oferta en el caso del productor cauto y optimista y demostramos que, aplicando el control sobre la cantidad de

productos ofertada, se puede evitar también un colapso económico, aunque esto es una propuesta más teórica que las anteriores.

Cuando la repetición es la mejor estrategia

En nuestras sociedades cada vez más interconectadas, debemos tomar decisiones muy complejas a diario, a veces debemos considerar simultáneamente muchos factores que están correlacionados, para resolver un problema muy simple. Tradicionalmente, los economistas neoclásicos utilizan en sus modelos un tipo de agentes llamado *Homo economicus*, en otras palabras, personas racionales con un poder computacional infinito que maximizan su función de utilidad utilizando toda la información presente en el sistema. Estos agentes son más parecidos a dioses que a la gente común. Pero esta visión simplista ha cambiado en el último medio siglo, gracias a las importantes contribuciones de algunos científicos que enfocaron sus estudios en analizar como realmente piensa la gente. Estos estudios han cambiado para siempre nuestra comprensión del proceso de toma de decisiones en situaciones inciertas. El paradigma del agente con racionalidad limitada es muy útil al diseñar mecanismos sociales, pero es muy difícil de integrar en los modelos económicos. Los programas informáticos pueden ayudarnos a resolver este problema, en particular los modelos basados en agentes. Uno de los ejemplos clásicos del uso de un modelo basado en agentes para estudiar un problema social complejo es el conocido *problema del Bar el Farol* introducido por el famoso economista irlandés Brian W. Arthur en 1994. Estudiamos una extensión de este problema que se lanzó en la primavera de 2018 por el Instituto Santa Fe, como una competición internacional en la cual participamos. A continuación, resumiremos el desarrollo de esta investigación y las principales conclusiones:

1. Desarrollamos un modelo basado en agentes en donde existen dos tipos de agentes: los tres grupos (pools) y los inversores. El objetivo de los inversores es maximizar su beneficio visitando los diferentes grupos (pools) utilizando alguna estrategia de acción. Desarrollamos 13 estrategias distintas y probamos su eficacia bajo muchas circunstancias, variando los principales parámetros del modelo. Agrupamos las distintas estrategias en cuatro familias de estrategias (ingenuo, aleatorio, agregado y basado en patrones) para estudiar cual es el mejor principio, para crear una estrategia ganadora.

2. Cuando existen en el sistema 50 inversores que utilizan diferentes estrategias, encontramos que la familia de estrategias que obtuvo el mejor resultado es la familia de estrategias ingenuas, en particular la estrategia en donde los inversores apostaron a lo largo de toda la simulación por el llamado grupo bajo (low pool). Esta estrategia ha demostrado su fortaleza a lo largo de todos los experimentos y en situaciones extremas. Calculamos las distribuciones del balance final de todas las estrategias que participaron repetitivamente en el experimento, y conclúimos que las estrategias: *siempre grupo bajo*, *siempre*

grupo alto y *secuencias de ceros* son las que ofrecen los mejores resultados en términos de balance final. Las distribuciones de cada estrategia son diferentes y como hemos mencionado anteriormente la estrategia *siempre grupo bajo* obtuvo la media más alta entre los tres grupos.

3. En términos de asistencia de inversores a cada uno de los grupos, encontramos la misma dinámica que descubrió Brian W. Arthur en su artículo original sobre el *Bar el Farol*. En nuestro caso, la asistencia en general converge en tan solo 10 iteraciones del modelo a los diferentes *atractores*. Después de repetir el experimento 100 veces y calcular la media de asistencia a cada grupo en cada iteración observamos que la asistencia media al grupo estable fue de 13.01 inversores, la asistencia al grupo bajo fue de 17.31 y la asistencia media al grupo alto fue de 19.16.

4. El efecto que tiene el coste de cambio de grupo (pool) τ sobre los balances finales de las diferentes estrategias es otro ejemplo de la fortaleza de la estrategia *siempre grupo bajo*, ya que es la única estrategia que al aumentar τ aumenta su balance final.

5. Investigamos qué pasa cuando aumentamos el número de inversores en el sistema gradualmente hasta 100. Como cabía esperar la estrategia *siempre grupo estable*, gracias a su esquema estable de reparto de recompensas es la única estrategia a la cual el aumento de inversores no afecta. Lo interesante es que también en este caso la estrategia *siempre grupo bajo* demuestra su gran fortaleza ya que mientras hayan menos de 73 inversores en el sistema esta estrategia consigue el balance total más alto.

Predicción en cuencas fractales

El oscilador de Duffing es un modelo de oscilador no lineal muy conocido y estudiado con múltiples aplicaciones en la modelización de muchos sistemas en la ciencia, la ingeniería y la economía. Es frecuentemente utilizado en la modelización de muelles no lineales, sistemas amortiguados o la dinámica de los mercados. Para algunos valores de los parámetros de este sistema aparece un espacio de fases fractalizado, que posee cuencas de *Wada*. Una cuenca de Wada es aquella para la cual toda su frontera está formada por puntos de Wada. Un punto de Wada pertenece a tres o más fronteras de distintas cuencas al mismo tiempo. La principal consecuencia de este hecho es la dificultad intrínseca para hacer predicciones, ya que una condición inicial ubicada en una frontera de Wada puede evolucionar temporalmente hacia cualquiera de los atractores. Esto tiene una gran importancia, ya que estamos acostumbrados a la idea del determinismo clásico, donde una vez fijamos la condición inicial, automáticamente conocemos la evolución de la órbita. Desde un punto de vista experimental, fijar una condición inicial con precisión infinita no es posible, de lo que se deriva un serio problema de predicción en los sistemas físicos y económicos. En este capítulo llevamos a cabo una profunda investigación estadística de este fenómeno y estos son los principales resultados:

1. El resultado principal de este trabajo de investigación, del cual, se derivan todos los demás, es el siguiente: aumentando la precisión de las condiciones iniciales o mejorando la resolución del espacio de fases hemos encontrado que la probabilidad de cada cuenca de atracción tiende a ser fija. Esto es resultado de la estructura fractal de las cuencas de atracción. Medimos la probabilidad de cada cuenca de atracción midiendo el área que ocupa en el espacio de fases. Asumiendo que esta área esta determinada por una frontera fractal nos lleva a la famosa paradoja de la línea de costa introducida por Mandelbrot. No importa cuánto mejoremos la medición de las condiciones iniciales obteniendo un espacio de fases con más resolución, la proporción de las cuencas de atracción se mantendrá fija. Utilizando una simulación de Monte Carlo confirmamos esta observación.

2. Hemos realizado un análisis unidimensional en donde medimos la probabilidad condicional de una condición inicial de pertenecer a una de las cuencas de atracción dependiendo de una de las coordenadas del espacio de fases. Encontramos que en la probabilidad condicional de pertenecer a alguna cuenca de atracción cambia drásticamente en las regiones del espacio de fases en donde se observan claramente los atractores, pero en regiones en donde se observa la propiedad de Wada (más comunes en el espacio de fases) la probabilidad condicional de pertenecer a alguna cuenca de atracción es bastante constante. Esto significa que, si conocemos una coordenada de la condición inicial, (es decir, tenemos en nuestro poder la mitad de la información relacionada con esa condición inicial) esta información no nos ayuda a hacer mejor las predicciones sobre el estado final del sistema.

3. Demostramos que, sabiendo únicamente las probabilidades de los atractores de un sistema dinámico, es suficiente para hacer predicciones también si el sistema posee la propiedad de Wada. Además, este método funciona también cuando no tenemos ninguna información de las condiciones iniciales.

4. Para investigar un sistema dinámico, no hace falta estudiar el espacio de fases con muy alta resolución. En nuestro caso una resolución de 300×300 capta todas las propiedades estadísticas globales del sistema. Esto es beneficioso en cuanto a tiempo de computo y recursos de computación.