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Approximation by polynomials in Sobolev spaces associated with classical moment functionals

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Abstract
Let $\mathbf{u}$ be a moment functional associated with the Hermite, Laguerre, or Jacobi classical orthogonal polynomials. We study approximation by polynomials in $H^r(\mathbf{u})$, the Sobolev space consisting of functions whose derivatives of consecutive orders up to r belong to the L^2 space associated with $\mathbf{u}$. This requires the simultaneous approximation of a function f and its consecutive derivatives up to order $N \leqslant r$. We explicitly construct orthogonal polynomials that achieve such simultaneous approximation and provide error estimates in terms of $E_n(f^{(r)})$, the error of best approximation of $f^{(r)}$ in $L^2(\mathbf{u})$.

Keywords Simultaneous approximation · Sobolev spaces · Linear functionals

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1 Introduction

In recent years, the study of orthogonal polynomials associated with Sobolev inner products (that is, inner products involving derivatives) have undergone intensive consideration. We refer the interested reader to the survey [28] for the latest presentation of the state of the art on Sobolev orthogonal polynomials. One of the most useful features of the Sobolev inner products is that they include terms "controlling" the behavior of the associated orthogonal polynomials on the boundary of the orthogonality domain.

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In particular, it is sometimes desirable for the corresponding orthogonal polynomials and their derivatives to have zeros on a set of points, for instance, on one or both end points of a closed interval $[a, b]$. This property is important in applications where, for a suitable function f, it is required that the partial sum of the corresponding Fourier orthogonal expansion and its derivatives up to some appropriate order $m \geqslant 1$, coincide with the values $f^{(j)}(a)$ and $f^{(j)}(b)$, $0 \leqslant j \leqslant m$ (e.g., solving boundary-value problems using spectral methods based on representing the solution in terms of Sobolev orthogonal polynomials, [6, 7, 14, 18, 19, 35]). The Sobolev inner products mostly studied in the literature to deal with the above problem are mainly of the form (see [12, 25, 30, 31, 34])

$$(f, g) = \sum_{j=0}^{m-1} \int_{-\infty}^{\infty} f^{(j)}(x)\, g^{(j)}(x)\, d\mu_j(x) + \int_{-1}^{1} f^{(m)}(x)\, g^{(m)}(x)\, \omega_{\alpha,\beta}(x)\, dx,$$

where μ_j, $0 \leqslant j \leqslant m-1$, are discrete Borel measures supported on $x = -1$ or $x = 1$, and $\omega_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is the Jacobi weight with $\alpha, \beta > -1$.

We note that orthogonal polynomials with respect to a weight function on $[a, b]$ (or, in general, on a bounded or unbounded interval I) lack the nice property of the zeros mentioned above ([33, Section 6.2]), and, moreover, their approximation behavior in Sobolev spaces give much weaker results than optimal. In spite of this, the approximating properties of Sobolev orthogonal polynomials is still insufficiently understood.

The purpose of this paper is to consider simultaneous approximation of a function and its derivatives by polynomials on an interval in L^2 norms associated with Jacobi, Laguerre, or Hermite weights. In order to explain our results, we need to introduce some notation.

For $n \geqslant 0$, let Π_n be the linear space of polynomials of a real variable and real coefficients of degree at most n, and let $\Pi = \bigcup_{n\geqslant 0} \Pi_n$.

Let Π^* denote the algebraic dual space of Π. That is, Π^* is the linear space of linear functionals defined on Π,

$$\Pi^* = \{\mathbf{u} : \Pi \to \mathbb{R} \quad \mathbf{u} \text{ is linear}\}.$$

We denote by $\langle \mathbf{u}, p\rangle$ the image of the polynomials p under the moment functional $\mathbf{u}$.

Any linear functional $\mathbf{u}$ is completely defined by the values

$$\mu_n := \langle \mathbf{u}, x^n\rangle, \quad n \geqslant 0,$$

and extended by linearity to all polynomials, where μ_n is called the n-th moment of $\mathbf{u}$. Therefore, we refer to $\mathbf{u}$ as a moment functional. $\mathbf{u}$ is called positive definite if $\langle \mathbf{u}, p^2\rangle > 0$ for every non-zero polynomial $p \in \Pi$. If $\mathbf{u}$ is positive definite, then there exists an *absolutely continuous measure* $d\mu$ supported in an infinite subset I or, equivalently, a real valued non-decreasing function $\varphi(x)$ such that ([9, Theorem 6.3])

$$\langle \mathbf{u}, p\rangle = \int_I p(x)\, d\mu = \int_I p(x)\, d\varphi(x).$$

In this paper we assume that **u** can be expressed as an integral with respect to a postive weight function $w(x)$ defined on an interval I, that is,

$$\langle \mathbf{u}, p\rangle = \int_I p(x) w(x) dx.$$

For a positive definite moment functional **u** associated with the weight function $w(x)$, let $L^2(\mathbf{u})$ be the linear space of functions given by

$$L^2(\mathbf{u}) = \left\{ \mathcal{C}_f : f \text{ is measurable and } \left\langle \mathbf{u}, f^2 \right\rangle < +\infty \right\},$$

where $\mathcal{C}_f$ is the class of functions equivalent to f in the following sense:

$$g \in \mathcal{C}_f \quad \text{if and only if} \quad \left\langle \mathbf{u}, (f-g)^2 \right\rangle = 0.$$

As usual, we will not distinguish between f and $\mathcal{C}_f$.

In $L^2(\mathbf{u})$ we can define the inner product

$$(f, g)_{\mathbf{u}} = \langle \mathbf{u}, f\, g\rangle, \quad f, g \in L^2(\mathbf{u}),$$

and the norm

$$\|f\|_{\mathbf{u}} = \sqrt{\langle \mathbf{u}, f^2\rangle}, \quad f \in L^2(\mathbf{u}).$$

The standard error of best approximation by polynomials of degree n is defined as

$$E_{n,\mathbf{u}}(f) := \inf_{p \in \Pi_n} \|f - p\|_{\mathbf{u}}.$$

For $r \geqslant 1$, let

$$H^r(\mathbf{u}) = \left\{ f \in L^2(\mathbf{u}) : \; f^{(m)} \in L^2(\mathbf{u}), \; 1 \leqslant m \leqslant r \right\}, \tag{1.1}$$

and we define the norm on $H^r(\mathbf{u})$ as

$$\|f\|_{H^r(\mathbf{u})} = \left(\sum_{m=0}^{r} \|f^{(m)}\|_{\mathbf{u}}^2 \right)^{1/2}.$$

In $L^2(\mathbf{u})$, the n-th partial sum of the Fourier orthogonal expansion $S_n f$ in terms of the orthogonal polynomials associated with **u** satisfies

$$E_{n,\mathbf{u}}(f) = \|f - S_n f\|_{\mathbf{u}}.$$

However, the approximating behavior of $S_n f$ in a Sobolev space is weaker. Indeed, we show that there is a non-zero polynomial ϕ of degree at most 2, such that for $f \in H^r(\mathbf{u})$ satisfying $\phi^{\frac{r-1}{2}} f^{(r)} \in L^2(\mathbf{u})$,

$$\| f - S_{2n} f \|_{H^r(\mathbf{u})} \leqslant c\, E_{n-1,\mathbf{u}} \left(\phi^{\frac{r-1}{2}} f^{(r)} \right), \tag{1.2}$$

where, hereon after, c denotes a generic positive constant independent of n and f, whose value may vary from line to line. On the other hand, we show that for $f \in H^r(\mathbf{u})$, there is a polynomial $p_n \in \Pi_{2n}$ satisfying

$$p_n^{(j)}(\nu_i) = f^{(j)}(\nu_i), \quad 0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s,$$

where $\nu_i \in \mathbb{R}$ are distinct and $d_1 + \cdots + d_s = r$, $d_i \in \mathbb{N}$, such that

$$\| f^{(m)} - p_n^{(m)} \|_{\mathbf{u}} \leqslant c\, n^{m-r+1}\, E_{n-r,\mathbf{u}} \left(f^{(r)} \right), \quad 0 \leqslant m \leqslant r, \tag{1.3}$$

for the Jacobi case, and

$$\| f^{(m)} - p_n^{(m)} \|_{\mathbf{u}} \leqslant c\, n^{(m-r+1)/2}\, E_{n-r,\mathbf{u}} \left(f^{(r)} \right), \quad 0 \leqslant m \leqslant r, \tag{1.4}$$

for the Laguerre and Hermite case. Evidently, (1.2) is weaker than (1.3) and (1.4), and holds under more restrictive conditions. Estimates of the form

$$\| f^{(m)} - p_n^{(m)} \|_{\mathbf{u}} \leqslant c\, n^{m-r}\, E_{n,\mathbf{u}} \left(f^{(r)} \right), \quad 0 \leqslant m \leqslant r, \tag{1.5}$$

have been established for the Jacobi case in [34]. We point out that our estimates (1.3) and (1.4) are coarser than (1.5) since our functional approach encompasses all three positive definite classical moment functionals at once. Therefore, finer estimates may be deduced by considering each case separately.

We give explicit expressions for the polynomial p_n in (1.3) and (1.4). In fact, it is the partial sum of the Fourier expansion in terms of orthogonal polynomials associated with a so-called *discrete–continuous* Sobolev inner product (see, for instance, [1, 12, 15, 17]). These polynomials are usually given in terms of classical orthogonal polynomials with non-standard parameters which require delicate extensions ([2, 3, 5, 14, 21, 23, 24, 29, 32]). We provide a more direct definition that holds for such non-standard parameters.

The paper is organized as follows. In the following section, we provide the basic facts about classical orthogonal polynomials needed to present our results. In Sect. 3, we discuss the approximation behavior of partial sums of Fourier expansions in terms of classical orthogonal polynomials, which give suboptimal results when used in Sobolev spaces. In Sect. 4, we present a discrete–continuous Sobolev inner product and construct associated orthogonal polynomials in terms of iterated integrals of classical orthogonal polynomials. The expressions of the Sobolev orthogonal polynomials provided there are suitable for studying Fourier expansions in terms of them. Simultaneous approximation by polynomials in $H^r(\mathbf{u})$ is studied in Sect. 5, and there we provide a more elaborate version of (1.3) and (1.4). A numerical example based on Laguerre polynomials is provided in the last section.

2 Classical orthogonal polynomials

In this section, we collect the basic facts about classical orthogonal polynomials that we will use throughout the work and which can be found in [13].

Let $\mathbf{u}$ be a moment functional. A sequence of polynomials $\{P_n(x)\}_{n\geqslant 0}$ is called an orthogonal polynomial sequence (OPS) with respect to $\mathbf{u}$ if

(1) $\deg\, P_n = n$,
(2) $\langle \mathbf{u}, P_n\, P_m\rangle = h_n\, \delta_{n,m}$, with $h_n \neq 0$.

Here $\delta_{n,m}$ denotes the Kronecker delta defined as

$$\delta_{n,m} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

If there is an OPS associated with $\mathbf{u}$, then $\mathbf{u}$ is called quasi-definite. Positive definite moment functionals are quasi-definite.

Observe that an OPS $\{P_n(x)\}_{n\geqslant 0}$ constitutes a basis for Π. If for all $n \geqslant 0$, the leading coefficient of $P_n(x)$ is 1, then $\{P_n(x)\}_{n\geqslant 0}$ is called a monic orthogonal polynomial sequence (MOPS).

Given a moment functional $\mathbf{u}$ and a polynomial $q(x)$, we define the left multiplication of $\mathbf{u}$ by $q(x)$ as the moment functional $q\,\mathbf{u}$ such that

$$\langle q\,\mathbf{u}, p\rangle \;=\; \langle \mathbf{u}, q\, p\rangle, \quad \forall p \in \Pi,$$

and we define the distributional derivative $D\mathbf{u}$ by

$$\langle D\mathbf{u}, p\rangle \;=\; -\langle \mathbf{u}, p'\rangle, \quad \forall p \in \Pi.$$

Moreover, the product rule is satisfied, that is,

$$D(q(x)\,\mathbf{u}) \;=\; q'(x)\,\mathbf{u} + q(x)\, D\mathbf{u}.$$

Definition 2.1 Let $\mathbf{u}$ be a quasi-definite moment functional, and let $\{P_n(x)\}_{n\geqslant 0}$ be an OPS with respect to $\mathbf{u}$. Then $\mathbf{u}$ is classical if there are nonzero polynomials $\phi(x)$ and $\psi(x)$ with $\deg\phi \leqslant 2$ and $\deg\psi = 1$, such that $\mathbf{u}$ satisfies the distributional Pearson equation

$$D(\phi\,\mathbf{u}) \;=\; \psi\,\mathbf{u}. \tag{2.1}$$

The sequence $\{P_n(x)\}_{n\geqslant 0}$ is called a classical OPS.

The following characterizations of classical moment functionals and OPS will be useful in the sequel.

Theorem 2.2 *Let* $\mathbf{u}$ *be a quasi-definite moment functional, and* $\{P_n(x)\}_{n\geqslant 0}$ *its associated MOPS. The following statements are equivalent:*

1. $\mathbf{u}$ *is a classical moment functional.*
2. *There are nonzero polynomials* $\phi(x)$ *and* $\psi(x)$ *with* $\deg\phi \leqslant 2$ *and* $\deg\psi = 1$ *such that, for* $n \geqslant 0$, $P_n(x)$ *satisfies*

$$\phi(x)\, P_n''(x) + \psi(x)\, P_n'(x) = \lambda_n\, P_n(x), \tag{2.2}$$

where $\lambda_n = n\,(\frac{n-1}{2}\phi'' + \psi')$.

3. *There is a nonzero polynomial* $\phi(x)$ *with* $\deg\phi \leqslant 2$, *such that* $\left\{\frac{P_{n+1}'(x)}{n+1}\right\}_{n\geqslant 0}$ *is the MOPS associated with the moment functional* $\mathbf{v} = \phi(x)\,\mathbf{u}$.
4. *There are real numbers* α_n *and* β_n, $n \geqslant 2$, *such that*

$$P_n(x) = \frac{P_{n+1}'(x)}{n+1} + \alpha_n \frac{P_n'(x)}{n} + \beta_n \frac{P_{n-1}'(x)}{n-1}, \quad n \geqslant 2. \tag{2.3}$$

It is well known (see [4] as well as [22]) that, up to affine transformations of the independent variable, the only families of positive definite classical orthogonal polynomials are the Hermite, Laguerre, and Jacobi polynomials. The corresponding moment functionals admit an integral representation of the form

$$\langle \mathbf{u}, p\rangle = \int_I p(x)\, w(x)\, dx, \quad p \in \Pi,$$

where $I = \mathbb{R}$ and $w(x) = e^{-x^2}$ in the Hermite case, $I = (0, +\infty)$ and $w(x) = x^\alpha e^{-x}$ with $\alpha > -1$ in the Laguerre case, and $I = (-1, 1)$ and $w(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$ in the Jacobi case. We note that in each case $w(x) > 0$ in I, and thus we say that $w(x)$ is a weight function.

In the sequel, we will need the explicit expression of the polynomials $\phi(x)$ and $\psi(x)$, and the parameters that appear in Theorem 2.2, as well as the square of the norms of the classical orthogonal polynomials, which we summarize in Table 1. We use [27] and [33] as reference.

Observe that from Theorem 2.2, if $\mathbf{u}$ is a classical moment functional satisfying (2.1), then $\mathbf{v} = \phi(x)\,\mathbf{u}$ is a classical moment functional satisfying the Pearson equation

$$D(\phi\,\mathbf{v}) = (\psi + \phi')\,\mathbf{v}.$$

We point out that if we consider $\phi(x)$ and $\psi(x)$ as in Table 1, then a positive definite classical moment functional $\mathbf{u}$ still satisfies (2.1) with $-\phi(x)$ and $-\psi(x)$. But then $(-\phi)^k\mathbf{u}$ is not necessarily a positive definite moment functional. In the sequel, we will only consider (2.1) with $\phi(x)$ given in Table 1 to guarantee the positive definiteness of $\mathbf{v} = \phi\,\mathbf{u}$.

Iterating this idea, we see that the high-order derivatives of classical orthogonal polynomials are again classical orthogonal polynomials of the same type.

Table 1 Explicit expressions of parameters in Theorem 2.2. Hereon, Γ denotes the Gamma function

$P_n(x)$	Hermite	Laguerre	Jacobi
$\phi(x)$	1	x	$1-x^2$
$\psi(x)$	$-2\,x$	$\alpha+1-x$	$\beta-\alpha-(\alpha+\beta+2)\,x$
λ_n	$-2\,n$	$-n$	$-n\,(n+\alpha+\beta+1)$
α_n	0	n	$\frac{2\,(\alpha-\beta)\,n}{(2n+\alpha+\beta)\,(2n+2+\alpha+\beta)}$
β_n	0	0	$\frac{-4\,(n-1)\,n\,(n+\alpha)\,(n+\beta)}{(2n+\alpha+\beta-1)\,(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
$\langle \mathbf{u}, P_n^2\rangle$	$\frac{n!}{2^n}\sqrt{\pi}$	$n!\,\Gamma(n+\alpha+1)$	$\frac{2^{\alpha+\beta+1}\,n!}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)^2}$

Theorem 2.3 *Let* $\mathbf{u}$ *be a classical moment functional satisfying* (2.1)*, and* $\{P_n(x)\}_{n\geqslant 0}$ *its corresponding MOPS. For* $k \geqslant 0$*, let* $\mathbf{v}_k = \phi^k(x)\,\mathbf{u}$ *and* $\{Q_{n,k}(x)\}_{n\geqslant 0}$ *be the sequence of polynomials given by*

$$Q_{n,k}(x) := \frac{1}{(n+k)!} P_{n+k}^{(k)}(x), \quad n \geqslant 0, \tag{2.4}$$

where $p^{(k)}$ *is the* k*-th derivative of* p*. Then, for each* $k \geqslant 0$*,* $\{Q_{n,k}(x)\}_{n\geqslant 0}$ *is an OPS associated with the moment functional* $\mathbf{v}_k$*, satisfying*

$$D(\phi\,\mathbf{v}_k) \;=\; \psi_k\,\mathbf{v}_k,$$

where $\psi_k(x) = \psi(x) + k\,\phi'(x)$*. Therefore,* $\mathbf{v}_k$ *is a classical moment functional.*

Clearly, the polynomials $\{Q_{n,k}(x)\}_{n\geqslant 0}$ defined in (2.4) are not monic. Nevertheless, they satisfy the following property: for $0 \leqslant r \leqslant n$, we have

$$Q_{n,k}^{(r)}(x) \;=\; Q_{n-r,k+r}(x). \tag{2.5}$$

Moreover, we can provide a distributional formulation of the differential equation (2.2) satisfied by these polynomials.

Proposition 2.4 *Let* $\mathbf{u}$ *be a classical functional satisfying* (2.1)*, and* $\{P_n(x)\}_{n\geqslant 0}$ *its corresponding MOPS. For* $k \geqslant 0$*, let* $\mathbf{v}_k = \phi^k(x)\,\mathbf{u}$*, and* $\{Q_{n,k}(x)\}_{n\geqslant 0}$ *be the sequence of polynomials defined in* (2.4)*. Then,*

$$D[\phi(x)\,Q_{n,k}'(x)\,\mathbf{v}_k] \;=\; \lambda_{n,k}\,Q_{n,k}(x)\,\mathbf{v}_k, \quad n \geqslant 0,$$

where $\lambda_{n,k} = (n+k)\,(\frac{n+k-1}{2}\phi'' + \psi') - k\,(\frac{k-1}{2}\phi'' + \psi')$*.*

Proof For $k \geqslant 0$, by Theorem 2.3, $\{Q_{n,k}(x)\}_{n\geqslant 0}$ is a sequence of classical OPS and $\mathbf{v}_k$ satisfies the Pearson equation $D(\phi\,\mathbf{v}_k) = \psi_k\,\mathbf{v}_k$ with $\psi_k(x) = \psi(x) + k\,\phi'(x)$.

Moreover, from Theorem 2.2, we have that $Q_{n,k}(x)$ satisfies the differential equation

$$\phi(x)\,Q_{n,k}''(x) + \psi_k(x)\,Q_{n,k}'(x) = \lambda_{n,k}\,Q_{n,k}(x), \quad n \geqslant 0,$$

with

$$\lambda_{n,k} = n\left(\frac{n-1}{2}\phi'' + \psi_k'\right) = (n+k)\left(\frac{n+k-1}{2}\phi'' + \psi'\right) - k\left(\frac{k-1}{2}\phi'' + \psi'\right).$$

Therefore, for $n \geqslant 0$,

$$\begin{aligned} D[\phi(x)\,Q_{n,k}'(x)\,\mathbf{v}_k] &= \phi(x)\,Q_{n,k}''(x)\,\mathbf{v}_k + Q_{n,k}'(x)\,D(\phi(x)\,\mathbf{v}_k) \\ &= \big(\phi(x)\,Q_{n,k}''(x) + \psi_k(x)\,Q_{n,k}'(x)\big)\,\mathbf{v}_k \\ &= \lambda_{n,k}\,Q_{n,k}(x)\,\mathbf{v}_k. \end{aligned}$$

□

Remark 2.5 We note that if $\{P_n(x)\}_{n\geqslant 0}$ is a sequence of Hermite or Laguerre polynomials, then

(1) for $k \geqslant 0$, $\{-\lambda_{n,k}\}_{n\geqslant 0}$ is an increasing sequence of non-negative real numbers (see Table 1),
(2) for $n \geqslant 0$ and $0 \leqslant i \leqslant k$, we have $-\lambda_{n,k} \leqslant -\lambda_{n+i,k-i}$.

In the Jacobi case, (1) and (2) hold when $\alpha + \beta + 2k \geqslant 0$.

3 Fourier series in terms of classical orthogonal polynomials

Let $\mathbf{u}$ be a positive definite classical moment functional satisfying (2.1), and $\{P_n(x)\}_{n\geqslant 0}$ its corresponding MOPS. For $k \geqslant 0$, let $\mathbf{v}_k = \phi^k(x)\,\mathbf{u}$, and $\{Q_{n,k}(x)\}_{n\geqslant 0}$ be the sequence of polynomials defined in (2.4). We note that $\mathbf{v}_0 = \mathbf{u}$, and $\mathbf{v}_k$ is a positive definite moment functional ([26, 27]).

For a function $f \in L^2(\mathbf{v}_k)$, we can define the Fourier coefficients of f as

$$\widehat{f}_{n,k} = \frac{\langle \mathbf{v}_k, f\, Q_{n,k}\rangle}{h_{n,k}} \quad \text{and} \quad h_{n,k} = \|Q_{n,k}\|^2_{\mathbf{v}_k}.$$

We note that $\langle \mathbf{v}_k, f\, Q_{n,k}\rangle$ is finite by virtue of the Cauchy-Schwarz inequality.

For $n, k \geqslant 0$, let $S_{n,k}$ denote the projection operator $S_{n,k} : L^2(\mathbf{v}_k) \to \Pi_n$ defined as

$$S_{n,k} f(x) = \sum_{j=0}^{n} \widehat{f}_{j,k}\, Q_{j,k}(x), \quad f \in L^2(\mathbf{v}_k).$$

This operator can be characterized in terms of $E_{n,\mathbf{v}_k}(f)$.

Theorem 3.1 *For $n, k \geqslant 0$ and $f \in L^2(\mathbf{v}_k)$, $S_{n,k} f(x)$ is the unique polynomial in Π_n satisfying*

$$E_{n,\mathbf{v}_k}(f) = ||f - S_{n,k} f||_{\mathbf{v}_k}.$$

Proof Since $\{Q_{n,k}(x)\}_{n\geqslant 0}$ is a basis of Π_n, we can write any polynomial $p \in \Pi_n$ as

$$p(x) = \sum_{j=0}^{n} c_j\, Q_{j,k}(x),$$

for some real coefficients c_j, $0 \leqslant j \leqslant n$. We compute

$$\frac{\partial}{\partial c_j}\|f - p\|^2_{\mathbf{v}_k} = \frac{\partial}{\partial c_j}\left(\|f\|^2_{\mathbf{v}_k} - 2\sum_{i=0}^{n} c_i \langle \mathbf{v}_k, f\, Q_{i,k}\rangle + \sum_{i=0}^{n} c_i^2\, h_{i,k}\right)$$

$$= -2\langle \mathbf{v}_k, f\, Q_{j,k}\rangle + 2\, c_j\, h_{j,k}.$$

To minimize $\|f-p\|_{\mathbf{v}_k}^2$ we impose $\dfrac{\partial}{\partial c_j}\|f-p\|_{\mathbf{v}_k}^2 = 0$ for $0 \leqslant j \leqslant n$. Since $\|f-p\|_{\mathbf{v}_k}^2$ is a convex function in the variables c_j, and taking into account that the Hessian matrix is positive definite, we deduce that the minimum of $\|f-p\|_{\mathbf{v}_k}^2$ is unique and is attained when $c_j = \widehat{f}_{j,k}$. □

It is well known (see [10, Chapter II, §4 – §8]) that for the Hermite, Laguerre, and Jacobi polynomials, we have

$$\lim_{n\to\infty} \|f - S_{n,k} f\|_{\mathbf{v}_k}^2 = 0, \quad f \in L^2(\mathbf{v}_k),$$

where $\mathbf{v}_k$ is the corresponding functional in each case. This means that we are allowed to write the Fourier expansion of $f \in L^2(\mathbf{v}_k)$,

$$f = \sum_{j=0}^{\infty} \widehat{f}_{j,k}\, Q_{j,k}(x),$$

where the equality holds for almost every point in the support I. By orthogonality, we obtain the Parseval identity

$$\|f\|_{\mathbf{v}_k}^2 = \sum_{j=0}^{\infty} \left|\widehat{f}_{j,k}\right|^2 h_{j,k}, \quad f \in L^2(\mathbf{v}_k).$$

It is possible to pass down relation (2.3) to the Fourier coefficients in the expansion of a function with respect to classical orthogonal polynomials.

Lemma 3.2 *Let* $\mathbf{u}$ *be a classical moment functional and* $\{P_n(x)\}_{n\geqslant 0}$ *be its associated MOPS. For* $n, k \geqslant 0$ *and* $f \in L^2(\mathbf{v}_k) \cap L^2(\mathbf{v}_{k+1})$,

$$\widehat{f}_{n,k+1} = \widehat{f}_{n,k} + \frac{\alpha_{n+k+1}}{n+k+1}\,\widehat{f}_{n+1,k} + \frac{\beta_{n+k+2}}{(n+k+1)\,(n+k+2)}\,\widehat{f}_{n+2,k},$$

where α_{n+k+1} *and* β_{n+k+2} *are the real numbers appearing in* (2.3).

Proof Differentiating (2.3) and using (2.4), we get $\{Q_{n,k}(x)\}_{n\geqslant 0}$ satisfies

$$Q_{n,k}(x) = Q_{n,k+1}(x) + \frac{\alpha_{n+k}}{n+k}\,Q_{n-1,k+1}(x) + \frac{\beta_{n+k}}{(n+k)\,(n+k-1)}\,Q_{n-2,k+1}(x). \tag{3.1}$$

Then, we compute

$$\langle \mathbf{v}_{k+1}, f\, Q_{n,k+1}\rangle = \left\langle \mathbf{v}_{k+1}, \left(\sum_{j=0}^{\infty} \widehat{f}_{j,k}\, Q_{j,k}\right) Q_{n,k+1}\right\rangle = \sum_{j=0}^{\infty} \widehat{f}_{j,k} \langle \mathbf{v}_{k+1}, Q_{j,k}\, Q_{n,k+1}\rangle.$$

Using (3.1), we obtain

$$\langle \mathbf{v}_{k+1}, f\, Q_{n,k+1}\rangle \\ = \left(\widehat{f}_{n,k} + \frac{\alpha_{n+1+k}}{n+k+1}\widehat{f}_{n+1,k} + \frac{\beta_{n+k+2}}{(n+k+1)\,(n+k+2)}\widehat{f}_{n+2,k}\right) h_{n,k+1},$$

and the desired result follows by dividing both sides by $h_{n,k+1}$. □

The following lemma will be useful often in the sequel.

Lemma 3.3 *For each $1 \leqslant r \leqslant k$ and $f \in L^2(\mathbf{v}_{k-r})$ such that $f^{(r)} \in L^2(\mathbf{v}_k)$,*

$$\widehat{f^{(r)}}_{n,k} = \widehat{f}_{n+r,k-r}. \tag{3.2}$$

Proof Using $Q'_{n+1,k-1}(x) = Q_{n,k}(x)$ and Proposition 2.4, we get

$$\begin{aligned}\left\langle \mathbf{v}_k,\, f^{(r)}\, Q_{n,k}\right\rangle &= -\left\langle D(Q'_{n+1,k-1}\,\phi\,\mathbf{v}_{k-1}),\; f^{(r-1)}\right\rangle \\ &= -\lambda_{n+1,k-1}\left\langle \mathbf{v}_{k-1},\; f^{(r-1)}\, Q_{n+1,k-1}\right\rangle.\end{aligned}$$

Then, iterating the result above $r-1$ more times, we obtain

$$\left\langle \mathbf{v}_k,\, f^{(r)}\, Q_{n,k}\right\rangle = \left((-1)^r \prod_{i=1}^{r} \lambda_{n+i,k-i}\right)\left\langle \mathbf{v}_{k-r},\, f\, Q_{n+r,k-r}\right\rangle, \tag{3.3}$$

where $\lambda_{n,k} = (n+k)\,(\frac{n+k-1}{2}\phi'' + \psi') - k\,(\frac{k-1}{2}\phi'' + \psi')$.

Setting $f(x) = Q_{n+r,k-r}(x)$ in (3.3), we get

$$\left\langle \mathbf{v}_k,\, Q^{(r)}_{n+r,k-r}\, Q_{n,k}\right\rangle = \left((-1)^r \prod_{i=1}^{r} \lambda_{n+i,k-i}\right)\left\langle \mathbf{v}_{k-r},\, Q^2_{n+r,k-r}\right\rangle.$$

Since $Q^{(r)}_{n+r,k-r}(x) = Q_{n,k}(x)$, we have

$$\left((-1)^r \prod_{i=1}^{r} \lambda_{n+i,k-i}\right) = \frac{\left\langle \mathbf{v}_k,\, Q^2_{n,k}\right\rangle}{\left\langle \mathbf{v}_{k-r},\, Q^2_{n+r,k-r}\right\rangle} = \frac{h_{n,k}}{h_{n+r,k-r}}. \tag{3.4}$$

Using (3.3) and the equation above, we obtain

$$\widehat{f^{(r)}}_{n,k} = \frac{\left\langle \mathbf{v}_k,\, f^{(r)}\, Q_{n,k}\right\rangle}{h_{n,k}} = \frac{\left\langle \mathbf{v}_{k-r},\, f\, Q_{n+r,k-r}\right\rangle}{h_{n+r,k-r}} = \widehat{f}_{n+r,k-r}.$$

□

It is possible to describe the behavior of the partial sum $S_{n,k} f$ under differentiation.

Proposition 3.4 *Let $n, k \geqslant 0$ and $0 \leqslant r \leqslant n$. If $f \in L^2(\mathbf{v}_k)$ such that $f^{(r)} \in L^2(\mathbf{v}_{k+r})$, then*

$$\left(S_{n,k} f\right)^{(r)} = S_{n-r,k+r} f^{(r)}. \tag{3.5}$$

Proof Using (2.5), we get

$$\left(S_{n,k} f\right)^{(r)} = \sum_{m=r}^{n} \widehat{f}_{m,k}\, Q_{m,k}^{(r)}(x) = \sum_{m=r}^{n} \widehat{f}_{m,k}\, Q_{m-r,k+r}(x) = \sum_{m=0}^{n-r} \widehat{f}_{m+r,k}\, Q_{m,k+r}(x).$$

From (3.2), we have $\widehat{f}_{m+r,k} = \widehat{f^{(r)}}_{m,k+r}$ and therefore $\left(S_{n,k} f\right)^{(r)} = S_{n-r,k+r} f^{(r)}$.

Now, we focus our attention on the approximation behavior of $S_{n,k} f$. First, we study the approximation in the Sobolev space

$$W_2^r(\mathbf{v}_k) = \left\{ f \in L^2(\mathbf{v}_k) : \; f^{(m)} \in L^2(\mathbf{v}_{k+m}), \; 1 \leqslant m \leqslant r \right\}, \quad r \geqslant 1.$$

Hence, for a function $f \in L^2(\mathbf{v}_k)$, we study the error of approximation for $f^{(m)}$ in $L^2(\mathbf{v}_{k+m})$.

Theorem 3.5 *Let $n, k \geqslant 0$, $r \geqslant 1$. For $f \in W_2^r(\mathbf{v}_k)$,*

$$\left\| f^{(m)} - (S_{n,k} f)^{(m)} \right\|_{\mathbf{v}_{k+m}} \leqslant (-\lambda_{n-r,k+r})^{(m-r)/2} E_{n-r,\mathbf{v}_{k+r}}\left(f^{(r)}\right), \quad 0 \leqslant m \leqslant r \leqslant n,$$

or, equivalently,

$$E_{n-m,\mathbf{v}_{k+m}}\left(f^{(m)}\right) \leqslant (-\lambda_{n-r,k+r})^{(m-r)/2} E_{n-r,\mathbf{v}_{k+r}}\left(f^{(r)}\right), \quad 0 \leqslant m \leqslant r \leqslant n, \tag{3.6}$$

holds for $k \geqslant 0$ in the Hermite case, for $\alpha + k > -1$ in the Laguerre case, and for $\alpha + k, \beta + k \geqslant 0$ in the Jacobi case.

Proof First, we prove (3.6) for $m = 0$. From the Parseval identity, (3.2) and the fact that $r \leq n$, we have

$$E_{n,\mathbf{v}_k}(f)^2 = \|f - S_{n,k} f\|_{\mathbf{v}_k}^2 = \sum_{j=n+1}^{\infty} \left|\widehat{f}_{j,k}\right|^2 \left\langle \mathbf{v}_k, Q_{j,k}^2 \right\rangle = \sum_{j=n+1}^{\infty} \left|\widehat{f^{(r)}}_{j-r,k+r}\right|^2 \left\langle \mathbf{v}_k, Q_{j,k}^2 \right\rangle,$$

and using (3.4), we get

$$E_{n,\mathbf{v}_k}(f)^2 = \sum_{j=n+1}^{\infty} \frac{\left|\widehat{f^{(r)}}_{j-r,k+r}\right|^2}{(-1)^r \prod_{i=1}^{r} \lambda_{j-r+i,k+r-i}} \left\langle \mathbf{v}_{k+r}, Q_{j-r,k+r}^2 \right\rangle.$$

By Remark 2.5, we have

$$(-1)^r \prod_{i=1}^{r} \lambda_{j-r+i,k+r-i} \geqslant (-\lambda_{j-r,k+r})^r.$$

Therefore,

$$E_{n,\mathbf{v}_k}(f)^2 \leqslant \sum_{j=n+1}^{\infty} \frac{\left|\widehat{f^{(r)}}_{j-r,k+r}\right|^2}{(-\lambda_{j-r,k+r})^r} \left\langle \mathbf{v}_{k+r}, Q^2_{j-r,k+r} \right\rangle,$$

holds for $k \geqslant 0$. Moreover, in any case $(-\lambda_{j-r,k+r})^r$ is an increasing sequence in j. Hence,

$$\begin{aligned} E_{n,\mathbf{v}_k}(f)^2 &\leqslant \frac{1}{(-\lambda_{n-r,k+r})^r} \sum_{j=n+1}^{\infty} \left|\widehat{f^{(r)}}_{j-r,k+r}\right|^2 \left\langle \mathbf{v}_{k+r}, Q^2_{j-r,k+r} \right\rangle \\ &= \frac{1}{(-\lambda_{n-r,k+r})^r} E_{n-r,\mathbf{v}_{k+r}} \left(f^{(r)}\right)^2, \quad 1 \leqslant r \leqslant n, \end{aligned} \tag{3.7}$$

and, thus, (3.6) holds for $m = 0$.

Now, let $m \geqslant 1$. From (3.5)

$$\left\| f^{(m)} - (S_{n,k} f)^{(m)} \right\|_{\mathbf{v}_{k+m}} = \left\| f^{(m)} - S_{n-m,k+m} f^{(m)} \right\|_{\mathbf{v}_{k+m}} = E_{n-m,\mathbf{v}_{k+m}} \left(f^{(m)}\right).$$

If we bound $E_{n-m,\mathbf{v}_{k+m}}\left(f^{(m)}\right)$ using (3.7) with r replaced with $r-m$, then we get

$$E_{n-m,\mathbf{v}_{k+m}} \left(f^{(m)}\right) \leqslant \frac{1}{(-\lambda_{n-r,k+r})^{(r-m)/2}} E_{n-r,\mathbf{v}_{k+r}} \left(f^{(r)}\right), \quad 1 \leqslant r \leqslant n,$$

which finishes the proof. □

In order to estimate the error of approximation of $S_{n,k}$ in $H^r(\mathbf{v}_k)$ defined in (1.1), we need the following lemma.

Lemma 3.6 *Let* $\mathbf{u}$ *be a classical moment functional and* $\{P_n(x)\}_{n\geqslant 0}$ *be its associated MOPS. For* $n, k \geqslant 0$,

$$Q_{n,k+1}(x) = \sum_{j=0}^{n} d^{(k)}_{n,j} \, Q_{j,k}(x),$$

with $d^{(k)}_{n,n} = 1$,

$$d^{(k)}_{n,n-1} = -\frac{\alpha_{n+k}}{n+k}, \quad d^{(k)}_{n,n-2} = -\left(\frac{\beta_{n+k}}{(n+k)(n+k-1)} - \frac{\alpha_{n+k}}{n+k}\frac{\alpha_{n+k-1}}{n+k-1}\right),$$

and

$$d_{n,j}^{(k)} = -\frac{\alpha_{n+k}}{n+k} d_{n-1,j}^{(k)} - \frac{\beta_{n+k}}{(n+k)\,(n+k-1)} d_{n-2,j}^{(k)}, \quad 0 \leqslant j \leqslant n-3, \quad n \geqslant 3.$$

where α_{n+k} and β_{n+k} are the real numbers appearing in (2.3).

Proof This result follows from iterating (3.1). □

It is obvious that, for Hermite polynomials, $Q_{n,k}(x) = Q_{n,0}(x)$ for $n, k \geqslant 0$. In the Laguerre case, it is straightforward to deduce from Table 1 the explicit expression for the coefficients in Lemma 3.6. For Jacobi polynomials, we can recover the expressions obtained in [34]. Indeed, for the Laguerre polynomials with parameter $\alpha > -1$, we obtain $d_{n,j}^{(k)} = (-1)^{n+j}$. We also recall here the coefficients for Jacobi polynomials with parameters $\alpha, \beta > -1$,

$$d_{n,j}^{(k)} = (-1)^{j+n} A_{j,k}^{\alpha,\beta} \, B_{n,k}^{\alpha,\beta} + A_{j,k}^{\beta,\beta} \, B_{n,k}^{\beta,\alpha},$$

where

$$A_{j,k}^{\alpha,\beta} = \frac{(\alpha+\beta+2k+2)_{2j}}{(\alpha+k+1)_j} \quad \text{and} \quad B_{n,k}^{\alpha,\beta} = \frac{(\alpha+k+1)_{n+1}}{(\alpha+\beta+2k+2)_{2n+1}}.$$

Our main effort lies in establishing the following result, which states the error of simultaneous approximation of a function and its derivatives by the projection operator $S_{n,k}$.

Theorem 3.7 *Let $k + r \geqslant 1$, and $f \in H^r(\mathbf{v}_k)$ such that $f' \in W_2^r(\mathbf{v}_{k+1})$. Then, for $n \geqslant r$,*

$$\left\| f^{(m)} - \left(S_{n,k} f\right)^{(m)} \right\|_{\mathbf{v}_k} \leqslant \frac{c\,(-\lambda_{n,k})^{m/2}}{(-\lambda_{n-r,k+r-1})^{(r-1)/2}} E_{n-r,\mathbf{v}_{k+r-1}} \left(f^{(r)}\right), \quad 0 \leqslant m \leqslant r, \tag{3.8}$$

holds for $k \geqslant 1$ in the Hermite case, for $\alpha + k > 0$ in the Laguerre case, and for $\alpha + k,\ \beta + k \geqslant 1$ in the Jacobi case.

Proof We shall consider each case separately.

First, let $\mathbf{u}$ be the moment functional associated with the Hermite polynomials. In this case, $\mathbf{u} = \mathbf{v}_k$ for $k \geqslant 0$, and thus $\|f\|_{\mathbf{u}} = \|f\|_{\mathbf{v}_k}$. It follows from (3.6) and the fact that in this case $\lambda_{n,k}$ is independent of the second subindex, that

$$\begin{aligned} \left\| f^{(m)} - \left(S_{n,k} f\right)^{(m)} \right\|_{\mathbf{v}_k} &\leqslant (-\lambda_{n-r,k+r})^{(m-r)/2} E_{n-r,\mathbf{v}_{k+r}} \left(f^{(r)}\right) \\ &\leqslant \frac{c\,(-\lambda_{n,k})^{m/2}}{(-\lambda_{n-r,k+r-1})^{(r-1)/2}} E_{n-r,\mathbf{v}_{k+r-1}} \left(f^{(r)}\right). \end{aligned}$$

For the Laguerre case, we start by considering the case $m = 0$. It is not difficult to modify the proof of Theorem 3.5 to deduce (3.8) since, by Remark 2.5, we have

$$(-1)^r \prod_{i=1}^{r} \lambda_{j-r+i,k+r-i} \geqslant (-\lambda_{j-r,k+r})^{r-1}.$$

Now, we consider the case $m = 1$. By triangle inequality

$$\left\| f' - \left(S_{n,k} f\right)' \right\|_{\mathbf{v}_k} \leqslant \| f' - S_{n-1,k} f' \|_{\mathbf{v}_k} + \left\| S_{n-1,k} f' - \left(S_{n,k} f\right)' \right\|_{\mathbf{v}_k}.$$

The first term on the right side is $E_{n-1,\mathbf{v}_k}(f')$. In order to bound the second term, consider the Fourier expansion of f'. Since $f \in H^r(\mathbf{v}_k)$ and, thus, $f' \in L^2(\mathbf{v}_k)$,

$$f' = \sum_{j=0}^{\infty} \widehat{f'}_{j,k}\, Q_{j,k}(x).$$

Moreover, since $f' \in W_2^r(\mathbf{v}_{k+1})$, which means that $f' \in L^2(\mathbf{v}_{k+1})$, we can also write

$$\begin{aligned} f' = \sum_{i=0}^{\infty} \widehat{f'}_{i,k+1}\, Q_{i,k+1}(x) &= \sum_{i=0}^{\infty} \sum_{j=0}^{i} (-1)^{i+j} \widehat{f}_{i+1,k} Q_{j,k}(x) \\ &= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} (-1)^{i+j} \widehat{f}_{i+1,k}\, Q_{j,k}(x). \end{aligned}$$

where the second equality follows from (3.2) and Lemma 3.6, and the third equality follows from interchanging the order of the summations. Comparing the two expressions for f', we deduce

$$\widehat{f'}_{j,k} = \sum_{i=j}^{\infty} (-1)^{i+j} \widehat{f}_{i+1,k}.$$

Using this, together with

$$\left(S_{n,k} f\right)' = \sum_{i=0}^{n-1} \widehat{f}_{i+1,k}\, Q_{i,k+1}(x) = \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} \widehat{f}_{i+1,k}\, Q_{j,k}(x),$$

we obtain

$$
\begin{aligned}
S_{n-1,k}f' - \left(S_{n,k}f\right)' &= \sum_{j=0}^{n-1}\left(\widehat{f'}_{j,k} - \sum_{i=j}^{n-1}(-1)^{i+j}\,\widehat{f}_{i+1,k}\right)Q_{j,k}(x)\\
&= \sum_{j=0}^{n-1}\sum_{i=n}^{\infty}(-1)^{i+j}\,\widehat{f}_{i+1,k}\,Q_{j,k}(x)\\
&= \widehat{f'}_{n,k}\sum_{j=0}^{n-1}Q_{j,k}(x).
\end{aligned}
$$

Therefore,

$$
\left\| S_{n-1,k}f' - \left(S_{n,k}f\right)'\right\|_{\mathbf{v}_k}^2 = \left|\widehat{f'}_{n,k}\right|^2 h_{n,k}\sum_{j=0}^{n-1}\frac{h_{j,k}}{h_{n,k}}.
$$

Using (3.4), we have

$$
h_{j,k} = \frac{(-1)^k\prod_{i=1}^{k}\lambda_{j+i,k-i}}{(j+k)!^2}\left\langle \mathbf{u},\, P_{j+k}^2\right\rangle. \tag{3.9}
$$

Moreover, by Table 1, and the fact that $-\lambda_{j+i,k-i} \leqslant -\lambda_{n+i,k-i}$ for $0 \leqslant j \leqslant n-1$, and $1 \leqslant i \leqslant k$, we can estimate

$$
\begin{aligned}
\sum_{j=0}^{n-1}\frac{h_{j,k}}{h_{n,k}} &= \sum_{j=0}^{n-1}\left(\frac{(n+k)!}{(j+k)!}\right)^2\frac{\prod_{i=1}^{k}\lambda_{j+i,k-i}}{\prod_{i=1}^{k}\lambda_{n+i,k-i}}\frac{\left\langle \mathbf{u},\, P_{j+k}^2\right\rangle}{\left\langle \mathbf{u},\, P_{n+k}^2\right\rangle}\\
&\leqslant \sum_{j=0}^{n-1}\frac{(n+k)!}{(j+k)!}\,\frac{(-\lambda_{j+k,0})^k}{(-\lambda_{n+1,k-1})^k}\,\frac{\Gamma(j+k+\alpha+1)}{\Gamma(n+k+\alpha+1)}\\
&\leqslant c\sum_{j=0}^{n-1}\frac{(-\lambda_{j+k,0})^k}{(-\lambda_{n+1,k-1})^k} \leqslant c\,(-\lambda_{n,k})
\end{aligned}
$$

Therefore, using (3.6), we obtain

$$
\left\| S_{n-1,k}f' - \left(S_{n,k}f\right)'\right\|_{\mathbf{v}_k} \leqslant c\,(-\lambda_{n,k})^{1/2}\,E_{n-1,\mathbf{v}_k}(f'),
$$

and, thus,

$$
\begin{aligned}
\left\| f' - \left(S_{n,k}f\right)'\right\|_{\mathbf{v}_k} &\leqslant E_{n-1,\mathbf{v}_k}(f') + c\,(-\lambda_{n,k})^{1/2}\,E_{n-1,\mathbf{v}_k}(f')\\
&\leqslant c\,(-\lambda_{n,k})^{1/2}\,E_{n-1,\mathbf{v}_k}(f').
\end{aligned}
$$

By (3.6), we get

$$\left\| f' - \left(S_{n,k} f\right)' \right\|_{\mathbf{v}_k} \leqslant \frac{c\,(-\lambda_{n,k})^{1/2}}{(-\lambda_{n-r,k+r-1})^{(r-1)/2}} E_{n-r,\mathbf{v}_{k+r-1}}\left(f^{(r)}\right).$$

This proves (3.8) in the Laguerre case for $m = 1$ and $r \geqslant 1$.

The case $m \geqslant 2$ follows inductively. With this in mind, we start by proving the following estimate

$$\left\| \left(S_{n-1,k} f' - (S_{n,k} f)'\right)^{(m)} \right\|_{\mathbf{v}_k}^2 \leqslant \frac{c\,(-\lambda_{n,k})^{m+1}}{(-\lambda_{n-r,k+r-1})^{r-1}} E_{n-r,\mathbf{v}_{k+r-1}}\left(f^{(r)}\right)^2. \quad (3.10)$$

Since

$$\left(S_{n-1,k} f' - (S_{n,k} f)'\right)^{(m)} = \widehat{f'}_{n,k} \sum_{j=m}^{n-1} Q_{j-m,k+m}(x),$$

then we estimate as follows

$$\begin{aligned}
\left\| \left(S_{n-1,k} f' - (S_{n,k} f)'\right)^{(m)} \right\|_{\mathbf{v}_k}^2 &= \left|\widehat{f'}_{n,k}\right|^2 \left\| \sum_{j=m}^{n-1} Q_{j-m,k+m} \right\|_{\mathbf{v}_k}^2 \\
&\leqslant \left|\widehat{f'}_{n,k}\right|^2 \, h_{n,k} \sum_{j=m}^{n-1} \frac{h_{j-m,k+m}}{h_{n,k}} \\
&= \left|\widehat{f'}_{n,k}\right|^2 \, h_{n,k} \sum_{j=m}^{n-1} \left((-1)^m \prod_{i=1}^{m} \lambda_{j-m+i,k+m-i} \right) \frac{h_{j,k}}{h_{n,k}} \\
&\leqslant \left|\widehat{f'}_{n,k}\right|^2 \, h_{n,k} \, (-\lambda_{n,k})^m \sum_{j=m}^{n-1} \frac{h_{j,k}}{h_{n,k}} \\
&\leqslant c\,(-\lambda_{n,k})^{m+1} \, E_{n-1,\mathbf{v}_k}(f')^2 \\
&\leqslant \frac{c\,(-\lambda_{n,k})^{m+1}}{(-\lambda_{n-r,k+r-1})^{r-1}} E_{n-r,\mathbf{v}_{k+r-1}}\left(f^{(r)}\right)^2,
\end{aligned}$$

where we have used (3.6) and (3.9). This establishes (3.10) for $m \geqslant 2$.

Assuming that (3.8) has been established for a fixed m, we prove that it also holds for $m + 1$. By triangle inequality,

$$\begin{aligned}
\left\| f^{(m+1)} - \left(S_{n,k} f\right)^{(m+1)} \right\|_{\mathbf{v}_k} \leqslant & \left\| \left(f' - S_{n-1,k} f'\right)^{(m)} \right\|_{\mathbf{v}_k} \\
& + \left\| \left(S_{n-1,k} f' - (S_{n,k} f)'\right)^{(m)} \right\|_{\mathbf{v}_k}.
\end{aligned}$$

The second term on the right side is bounded in (3.10). The first term on the right side can be bounded, by induction with r replaced with $r-1$, by a bound that is less than the above. This completes the proof of the Laguerre case.

The Jacobi case has already been established in [34] in the form

$$\left\| f^{(m)} - \left(S_{n,k} f\right)^{(m)} \right\|_{\mathbf{v}_k} \leqslant c\, n^{-r+2m-1/2}\, E_{n-r,\mathbf{v}_k}\left(f^{(r)}\right), \quad 0 \leqslant m \leqslant r.$$

Here, we recast the important points of the proof in [34] in terms of $\lambda_{n,k}$ and $E_{n-r,\mathbf{v}_{k+r-1}}(f^{(r)})$. Since in the Jacobi case $-\lambda_{n,k} \sim n^2$, then for $m=1$ and $r=1$, we obtain

$$\left\| f' - \left(S_{n,k} f\right)' \right\|_{\mathbf{v}_k} \leqslant c\, n^{1/2}\, E_{n-1,\mathbf{v}_k}\left(f'\right) \leqslant c\,(-\lambda_{n,k})^{1/2}\, E_{n-1,\mathbf{v}_k}\left(f'\right).$$

Then, (3.8) is established for $m=1$ and $r \geqslant 1$ by using (3.6). Moreover, for $m \geqslant 1$, from (2.18) in [34] we have

$$\begin{aligned} \left\| \left(S_{n-1,k} f' - (S_{n,k} f)'\right)^{(m)} \right\|_{\mathbf{v}_k} &\leqslant c\,(-\lambda_{n,k})^{m+1/2} E_{n-1,\mathbf{v}_k}(f') \\ &\leqslant c\,(-\lambda_{n,k})^{m+1}\, E_{n-1,\mathbf{v}_k}(f'). \end{aligned}$$

Then (3.8) follows from (3.6) and by induction as in the Laguerre case. □

4 Discrete–continuous Sobolev orthogonal polynomials

This section is devoted to studying sequences of orthogonal polynomials with respect to a discrete–continuous Sobolev inner product associated with a positive definite classical moment functional $\mathbf{u}$ and a set of s distinct real numbers

$$\nu_1 < \nu_2 < \cdots < \nu_s.$$

More concretely, the inner product that we consider is of the form

$$(f, g) = (F_1^\top, \ldots, F_s^\top)\, \Lambda \begin{pmatrix} G_1 \\ \vdots \\ G_s \end{pmatrix} + \left\langle \mathbf{u}, f^{(N)} g^{(N)} \right\rangle, \tag{4.1}$$

where

$$F_i = \left(f(\nu_i),\ f'(\nu_i), \ldots, \frac{f^{(j)}(\nu_i)}{j!}, \ldots, \frac{f^{(d_i-1)}(\nu_i)}{(d_i-1)!} \right)^\top, \quad 1 \leqslant i \leqslant s, \tag{4.2}$$

with $d_1 + \cdots + d_s = N$, $d_i \in \mathbb{N}$, and Λ is a $N \times N$ positive definite symmetric matrix.

In this section, we present two main results:

1. For a given sequence of polynomials (defined below in (4.6)), there is a matrix Λ such that the sequence of polynomials is orthogonal with respect to (4.1).
2. For a given matrix Λ, there is a sequence of orthogonal polynomials associated with (4.1).

In order to prove these results, we need to introduce a basis of polynomials and its dual basis associated with the set $\{\nu_i : 1 \leqslant i \leqslant s\}$.

We define the following polynomial associated with (4.1),

$$\varphi(x) = \prod_{i=1}^{s} (x - \nu_i)^{d_i}.$$

Hence, deg $\varphi = N$. In the sequel, we fix $\theta \in [\nu_1, \nu_s]$. We introduce the basis of Π,

$$\mathfrak{B}_{\theta,\varphi} = \{(x-\theta)^m \varphi^n(x) : \quad n \geqslant 0, \quad 0 \leqslant m \leqslant N-1\}, \tag{4.3}$$

and its associated dual basis;

$$\mathfrak{B}'_{\theta,\varphi} = \{\boldsymbol{\sigma}_{m,n} : \quad n \geqslant 0, \quad 0 \leqslant m \leqslant N-1\}, \tag{4.4}$$

such that $\langle \boldsymbol{\sigma}_{i,j}, (x-\theta)^m \varphi^n \rangle = \delta_{i,m}\delta_{j,n}$. Notice that for every $p \in \Pi$,

$$p(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{N-1} a_{m,n}\, (x-\theta)^m\, \varphi^n(x), \quad a_{m,n} = \langle \boldsymbol{\sigma}_{m,n}, p \rangle,$$

where $a_{m,n} = 0$ if $m + nN > \deg p$.

Let

$$\ell_{i,j}(x) = \varphi(x) \frac{(x-\nu_i)^{j-d_i}}{j!} \frac{d^{d_i-j-1}}{dx^{d_i-j-1}} \left[\frac{(x-\nu_i)^{d_i}}{\varphi(x)} \right] \Bigg|_{x=\nu_i},$$

$$0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s.$$

For a differentiable function $f(x)$, the Hermite interpolation polynomial $\mathcal{H}_\varphi f(x)$ defined as ([11])

$$\mathcal{H}_\varphi f(x) = \sum_{i=1}^{s} \sum_{j=0}^{d_i-1} f^{(j)}(\nu_i)\, \ell_{i,j}(x),$$

is the unique polynomial of degree at most $N-1$ such that

$$\left(\mathcal{H}_\varphi f\right)^{(j)}(\nu_i) = f^{(j)}(\nu_i), \quad 0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s. \tag{4.5}$$

We will need to extend the polynomials defined in (2.4) to negative values of the parameter k. Let $\{Q_{n,0}(x)\}_{n\geqslant 0}$ be the polynomials defined in (2.4) associated with the classical moment functional $\mathbf{u}$. For $N \geqslant 1$, we define recursively the polynomial

$$Q_{0,-N}(x) = 1, \quad Q_{n,-N}(x) = \int_{\theta}^{x} Q_{n-1,-N+1}(t)dt, \quad n \geqslant 1.$$

Observe that $\deg Q_{n,-N} = n$, and $Q^{(r)}_{n,-N}(x) = Q_{n-r,-N+r}(x)$ for $0 \leqslant r \leqslant n$. Moreover,

$$Q^{(r)}_{n,-N}(\theta) = 0, \quad 0 \leqslant r \leqslant N-1.$$

In this way, by Taylor's theorem, we have the following alternative expression for $Q_{n,-N}(x)$ with $1 \leqslant N \leqslant n$,

$$Q_{n,-N}(x) = \int_{\theta}^{x} \frac{(x-t)^{N-1}}{(N-1)!} Q_{n-N,0}(t)\, dt.$$

We are ready to state our first result.

Theorem 4.1 *There is a $N \times N$ positive definite symmetric matrix Λ such that the set of polynomials $\{q_n(x)\}_{n\geqslant 0}$ with*

$$q_n(x) = \begin{cases} (x-\theta)^n, & 0 \leqslant n \leqslant N-1, \\ Q_{n,-N}(x) - \mathcal{H}_{\varphi} Q_{n,-N}(x), & n \geqslant N, \end{cases} \tag{4.6}$$

is orthogonal with respect to (4.1)*. Moreover,*

$$(q_n, q_n) = 1, \quad 0 \leqslant n \leqslant N-1, \quad \textit{and} \quad (q_n, q_n) = h_{n-N,0}, \quad n \geqslant N.$$

Proof First, we prove the existence of a $N \times N$ non-singular matrix M such that, for all polynomials $p(x)$ and $q(x)$,

$$(P_1^{\top}, \ldots, P_s^{\top})\,(M^{-1})^{\top} M^{-1} \begin{pmatrix} Q_1 \\ \vdots \\ Q_s \end{pmatrix} = \sum_{j=0}^{N-1} \frac{p^{(j)}(\theta)}{j!}\,\frac{q^{(j)}(\theta)}{j!}. \tag{4.7}$$

Then we show that $\{q_n(x)\}_{n\geqslant 0}$ is orthogonal with respect to (4.1) with $\Lambda = (M^{-1})^{\top} M^{-1}$ which is evidently a positive definite symmetric matrix ([20]).

Using the bases $\mathfrak{B}_{\theta,\varphi}$ and $\mathfrak{B}'_{\theta,\varphi}$ introduced in (4.3) and (4.4), for each $1 \leqslant i \leqslant s$, we write

$$\frac{(-1)^j}{j!}\boldsymbol{\delta}^{(j)}(x-\nu_i) = \sum_{n=0}^{+\infty}\sum_{m=0}^{N-1} c^{(i,j)}_{m,n}\, \boldsymbol{\sigma}_{m,n},$$

where

$$c^{(i,j)}_{m,n} = \left\langle \frac{(-1)^j}{j!}\boldsymbol{\delta}^{(j)}(x-\nu_i),\ (x-\theta)^m \varphi^n \right\rangle.$$

For $n \geqslant 1$, the polynomial $\varphi(x)^n$ can be represented as

$$\varphi(x)^n = (x-\nu_i)^{nd_i}\rho_i(x), \quad \text{where} \quad \rho_i(\nu_i) \neq 0.$$

Then,

$$\frac{d^j}{dt^j}\varphi(x)^n = \sum_{k=0}^{j} \frac{j!}{k!(j-k)!}\frac{(nd_i)!}{(nd_i-k)!}(x-\nu_i)^{nd_i-k}\rho_i^{(j-k)}(x).$$

For $0 \leqslant j \leqslant d_i - 1$, it follows that

$$c_{m,n}^{(i,j)} = 0, \qquad n \geqslant 1,$$

which means that

$$\frac{(-1)^j}{j!}\boldsymbol{\delta}^{(j)}(x-\nu_i) = \sum_{m=j}^{N-1}\binom{m}{j}(\nu_i-\theta)^{m-j}\,\boldsymbol{\sigma}_{m,0}, \qquad 0 \leqslant j \leqslant d_i - 1.$$

Given any polynomial $p(x)$, for each $1 \leqslant i \leqslant s$, we apply both sides of the above distributional equations to $p(x)$ and obtain

$$\frac{p^{(j)}(\nu_i)}{j!} = \sum_{m=j}^{N-1}\binom{m}{j}(\nu_i-\theta)^{m-j}\langle\boldsymbol{\sigma}_{m,0},\ p\rangle, \qquad 0 \leqslant j \leqslant d_i - 1.$$

Define the $d_i \times N$ matrices

$$M_i = \begin{pmatrix} 1 & (\nu_i-\theta) & (\nu_i-\theta)^2 & (\nu_i-\theta)^3 & \ldots\ldots & (\nu_i-\theta)^{N-1} \\ & 1 & 2(\nu_i-\theta) & 3(\nu_i-\theta)^2 & \ldots\ldots & (N-1)(\nu_i-\theta)^{N-2} \\ & & 1 & 3(\nu_i-\theta) & \ldots\ldots & \binom{N-1}{2}(\nu_i-\theta)^{N-3} \\ & & & \ddots & \ddots \ldots & \vdots \\ & & & & 1 \;\ldots & \binom{N-1}{d_i-1}(\nu_i-\theta)^{N-d_i} \end{pmatrix}. \tag{4.8}$$

For each $1 \leqslant i \leqslant s$, we have

$$P_i = M_i \begin{pmatrix} \langle\boldsymbol{\sigma}_{0,0},\ p\rangle \\ \vdots \\ \langle\boldsymbol{\sigma}_{N-1,0},\ p\rangle \end{pmatrix},$$

where P_i is the vector defined in (4.2), and, as a consequence,

$$\begin{pmatrix} P_1 \\ \vdots \\ P_s \end{pmatrix} = \begin{pmatrix} M_1 \\ \vdots \\ M_s \end{pmatrix}\begin{pmatrix} \langle\boldsymbol{\sigma}_{0,0},\ p\rangle \\ \vdots \\ \langle\boldsymbol{\sigma}_{N-1,0},\ p\rangle \end{pmatrix} = M\begin{pmatrix} \langle\boldsymbol{\sigma}_{0,0},\ p\rangle \\ \vdots \\ \langle\boldsymbol{\sigma}_{N-1,0},\ p\rangle \end{pmatrix}. \tag{4.9}$$

We now show that the $N \times N$ matrix M is non-singular. By construction, (4.9) has at least one solution $(\langle \boldsymbol{\sigma}_{0,0}, p\rangle, \dots, \langle \boldsymbol{\sigma}_{N-1,0}, p\rangle)^\top$. Suppose that $(g_0, \dots, g_{N-1})^\top$ is another solution of (4.9). Define the polynomials

$$f(x) = \sum_{m=0}^{N-1} \langle \boldsymbol{\sigma}_{m,0},\ p\rangle\ (x-\theta)^m \quad \text{and} \quad g(x) = \sum_{m=0}^{N-1} g_m\ (x-\theta)^m.$$

Then, for each $1 \leqslant i \leqslant s$,

$$f^{(j)}(\nu_i) = \frac{p^{(j)}(\nu_i)}{j!} = g^{(j)}(\nu_i), \qquad 0 \leqslant j \leqslant d_i - 1.$$

Let $h(x) = f(x) - g(x)$. Clearly $\deg h \leqslant N-1$. On the other hand,

$$h^{(j)}(\nu_i) = 0, \qquad 0 \leqslant j \leqslant d_i - 1.$$

This implies that ν_i is a zero of $h(t)$ with multiplicity at least d_i. But this is true for $1 \leqslant i \leqslant s$. It follows that $\deg h \geqslant N$. Therefore, $h(x) = 0$, or, equivalently, $g_m = \langle \boldsymbol{\sigma}_{m,0}, p\rangle$ for $0 \leqslant m \leqslant N-1$. Hence, M is a non-singular symmetric matrix.

Now, we prove the orthogonality of $\{q_n(x)\}_{n\geqslant 0}$ with respect to (4.1) with $\Lambda = (M^{-1})^\top M^{-1}$. For $0 \leqslant n \leqslant N-1$, $q_n^{(N)}(x) = 0$. Then, by (4.7),

$$(q_n, q_m) = \sum_{j=0}^{N-1} \frac{q_n^{(j)}(\theta)}{j!} \frac{q_m^{(j)}(\theta)}{j!}.$$

If $0 \leqslant m \leqslant N-1$, then it is clear that $(q_n, q_m) = \delta_{n,m}$. For $m \geqslant N$, by (4.5), $(q_n, q_m) = 0$. Furthermore, for $n, m \geqslant N$,

$$(q_n, q_m) = \left\langle \mathbf{u}, q_n^{(N)}\, q_m^{(N)} \right\rangle = \langle \mathbf{u}, Q_{n-N,0}\, Q_{m-N,0}\rangle = h_{n-N,0}\, \delta_{n,m}.$$

$\square$

Remark 4.2 It is important to note that (4.9) is equivalent to

$$\begin{pmatrix} 1 \\ x-\theta \\ \vdots \\ (x-\theta)^{N-1} \end{pmatrix} = M^\top \begin{pmatrix} \ell_{1,0}(x) \\ \vdots \\ \ell_{1,d_1-1}(x) \\ \vdots \\ \ell_{s,0}(x) \\ \vdots \\ \ell_{s,d_s-1}(x) \end{pmatrix}. \tag{4.10}$$

where the matrix M was defined in (4.8)–(4.9). Furthermore, let $(\cdot,\cdot)_{\Pi_{N-1}}$ denote the restriction of (4.1) to Π_{N-1}. Then Λ is the Gram matrix of $(\cdot,\cdot)_{\Pi_{N-1}}$ with respect to the basis $\{\ell_{i,j}(x) :\ 0 \leqslant j \leqslant d_i - 1,\ 1 \leqslant i \leqslant s\}$ and, thus,

$$G := M^\top \Lambda M,$$

is the Gram matrix of $(\cdot,\cdot)_{\Pi_{N-1}}$ with respect to the basis $\{(x-\theta)^m :\ 0 \leqslant m \leqslant N-1\}$.

Now we deal with the case when Λ in (4.1) is a prescribed $N \times N$ positive definite symmetric matrix. Recall that in this case there is a unique lower triangular matrix Ξ with positive real numbers in the main diagonal, such that $\Lambda = \Xi\, \Xi^\top$ ([20]). In this way,

$$G = M^\top \Lambda M = M^\top \Xi\, \Xi^\top M.$$

Since the factor $M^\top \Xi$ is a non-singular matrix, we conclude that G is a positive definite matrix ([20]). We denote by L^{-1} the unique lower triangular matrix with positive real numbers in the main diagonal such that $G = L^{-1}(L^{-1})^\top$.

In this way, we can state our second main result in terms of Λ and L^{-1}.

Theorem 4.3 *For a $N \times N$ positive definite symmetric matrix Λ, the sequence of polynomials $\{\widetilde{q}_n(x)\}_{n\geqslant 0}$ given by*

$$\begin{pmatrix} \widetilde{q}_0(x) \\ \widetilde{q}_1(x) \\ \vdots \\ \widetilde{q}_{N-1}(x) \end{pmatrix} = L \begin{pmatrix} 1 \\ x-\theta \\ \vdots \\ (x-\theta)^{N-1} \end{pmatrix}, \tag{4.11}$$

and,

$$\widetilde{q}_n(x) = Q_{n,-N}(x) - \mathcal{H}_\varphi Q_{n,-N}(x), \quad n \geqslant N,$$

is orthogonal with respect to (4.1) with Λ. Moreover,

$$(\widetilde{q}_n, \widetilde{q}_n) = 1, \quad 0 \leqslant n \leqslant N-1, \quad \textit{and} \quad (\widetilde{q}_n, \widetilde{q}_n) = h_{n-N,0}, \quad n \geqslant N.$$

Proof Consider the matrix

$$H = \begin{pmatrix} (\widetilde{q}_0,\widetilde{q}_0) & (\widetilde{q}_0,\widetilde{q}_1) & \ldots & (\widetilde{q}_0,\widetilde{q}_{N-1}) \\ (\widetilde{q}_1,\widetilde{q}_0) & (\widetilde{q}_1,\widetilde{q}_1) & \ldots & (\widetilde{q}_1,\widetilde{q}_{N-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\widetilde{q}_{N-1},\widetilde{q}_0) & (\widetilde{q}_{N-1},\widetilde{q}_1) & \ldots & (\widetilde{q}_{N-1},\widetilde{q}_{N-1}) \end{pmatrix}.$$

Note that by (4.11), $H = L\,G\,L^\top$. Therefore, H is the identity matrix of order N. It follows that

$$(\widetilde{q}_n, \widetilde{q}_m) = \delta_{n,m}, \quad 0 \leqslant n, m \leqslant N-1.$$

For $n \geqslant N$, it is clear that $(\widetilde{q}_n, \widetilde{q}_m) = h_{n-N,0}\,\delta_{n,m}$, $m \geqslant 0$. □

Remark 4.4 It is important to note that L and L^{-1} are both lower triangular matrices with non-zero real numbers in the main diagonal. Consequently, for $\widetilde{q}_n(x)$ defined in (4.11), we have $\deg \widetilde{q}_n = n$.

5 Fourier series in terms of Sobolev orthogonal polynomials

In this section, we study the approximation behavior of the Fourier series associated with the two sequences of orthogonal polynomials (4.6) and (4.11) introduced in the previous section.

Let $\mathbf{u}$ be a positive definite classical moment functional. For $f \in H^N(\mathbf{u})$, we define the Fourier orthogonal expansion of f with respect to the orthogonal basis (4.6) as,

$$f = \sum_{n=0}^{\infty} \mathfrak{f}_n^N \, q_n(x), \quad \mathfrak{f}_n^N = \frac{(f, q_n)}{(q_n, q_n)},$$

and with respect to the orthogonal basis (4.11) as,

$$f = \sum_{n=0}^{\infty} \widetilde{\mathfrak{f}}_n^N \, \widetilde{q}_n(x), \quad \widetilde{\mathfrak{f}}_n^N = \frac{(f, \widetilde{q}_n)}{(\widetilde{q}_n, \widetilde{q}_n)}.$$

For $n \geqslant 0$, let $\mathcal{S}_n^N$ and $\widetilde{\mathcal{S}}_n^N$ denote the projection operators $\mathcal{S}_n^N : H^N(\mathbf{u}) \to \Pi_n$ and $\widetilde{\mathcal{S}}_n^N : H^N(\mathbf{u}) \to \Pi_n$ defined as

$$\mathcal{S}_n^N f(x) = \sum_{j=0}^{n} \mathfrak{f}_j^N \, q_j(x) \quad \text{and} \quad \widetilde{\mathcal{S}}_n^N f(x) = \sum_{j=0}^{n} \widetilde{\mathfrak{f}}_j^N \, \widetilde{q}_j(x).$$

For $N = 0$, the operators $\mathcal{S}_n^0 f(x) = \widetilde{\mathcal{S}}_n^0 f(x) = S_{n,0} f(x)$ is the partial sum of the usual classical expansion in orthogonal polynomials. These operators satisfy several simple properties and can be written, in particular, in terms of the partial sum $S_{n-N,0} f$.

Lemma 5.1 *For $f \in H^N(\mathbf{u})$,*

(1) $\mathfrak{f}_n^N = f^{(n)}(\theta)/n!$ if $0 \leqslant n \leqslant N-1$, and $\mathfrak{f}_n^N = \widehat{f^{(N)}}_{n-N,0}$ if $n \geqslant N$;
(2) $\left(\mathcal{S}_n^N f\right)^{(N)} = S_{n-N,0} f^{(N)}$ if $n \geqslant N$;
(3) for $n \geqslant N$,

$$\left(\mathcal{S}_n^N f\right)^{(j)} (\nu_i) = f^{(j)}(\nu_i), \quad 0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s.$$

Proof As in the proof of Theorem 4.1, it is easy to see that if $0 \leqslant n \leqslant N-1$, $(f, q_n) = f^{(n)}(\theta)/n!$ and $(q_n, q_n) = 1$, whereas if $n \geqslant N$,

$$(f, q_n) = \left\langle \mathbf{u}, f^{(N)} \, Q_{n,-N}^{(N)} \right\rangle = \left\langle \mathbf{u}, f^{(N)} \, Q_{n-N,0} \right\rangle,$$

and $(q_n, q_n) = h_{n-N,0}$. In this way, (1) is proved. The statement (2) follows easily from (1).

Now, by (4.9) and (4.10),

$$\sum_{m=0}^{N-1} \frac{f^{(m)}(\theta)}{m!}(x-\theta)^m = \left(f(\theta), f'(\theta), \ldots, \frac{f^{(N-1)}(\theta)}{(N-1)!} \right) M^\top (M^{-1})^\top \begin{pmatrix} 1 \\ x-\theta \\ \vdots \\ (x-\theta)^{N-1} \end{pmatrix}$$

$$= \mathcal{H}_\varphi f(x),$$

where M is the matrix defined in (4.9). Therefore,

$$\mathcal{S}_n^N f(x) = \mathcal{H}_\varphi f(x) + \sum_{j=N}^{n} \mathfrak{f}_j^N q_j(x), \quad n \geqslant N.$$

Moreover, by (4.6), if $n \geqslant N$, $q_n^{(j)}(\nu_i) = 0$, $0 \leqslant j \leqslant d_i - 1$, $1 \leqslant i \leqslant s$, and, thus

$$\left(\mathcal{S}_n^N f\right)^{(j)}(\nu_i) = \left(\mathcal{H}_\varphi f\right)^{(j)}(\nu_i) = f^{(j)}(\nu_i), \quad 0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s.$$

This proves (3). □

Lemma 5.2 *For $f \in H^N(\mathbf{u})$,*

(1) $\widetilde{\mathfrak{f}}_n^N = \widehat{f^{(N)}}_{n-N,0}$ if $n \geqslant N$, and

$$\left(\widetilde{\mathfrak{f}}_0^N, \ldots, \widetilde{\mathfrak{f}}_{N-1}^N\right) = \left(f(\theta), f'(\theta), \ldots, \frac{f^{(N-1)}(\theta)}{(N-1)!} \right) L^{-1};$$

(2) $\left(\widetilde{\mathcal{S}}_n^N f\right)^{(N)} = S_{n-N,0} f^{(N)}$ if $n \geqslant N$;
(3) for $n \geqslant N$,

$$\left(\widetilde{\mathcal{S}}_n^N f\right)^{(j)}(\nu_i) = f^{(j)}(\nu_i), \quad 0 \leqslant j \leqslant d_i - 1, \quad 1 \leqslant i \leqslant s.$$

Proof It is easy to see that (1) follows from (4.11) and that (2) follows from (1). The proof of (3) is as in Lemma 5.1 after noticing that by (1) and (4.11), we have

$$\left(\widetilde{\mathfrak{f}}_0^N, \ldots, \widetilde{\mathfrak{f}}_{N-1}^N\right) \begin{pmatrix} q_0(x) \\ \vdots \\ q_{N-1}(x) \end{pmatrix} = \sum_{m=0}^{N-1} \frac{f^{(m)}(\theta)}{m!}(x-\theta)^m.$$

□

Now we consider the simultaneous approximation by polynomials. First, we establish estimates for derivatives of order at least N.

Theorem 5.3 *Let $r \geqslant N+1$ and $f \in H^r(\mathbf{u})$ such that $f^{(r)} \in L^2(\mathbf{v}_{r-N-1})$. Then*

$$\left\| f^{(m)} - \left(\mathcal{S}_n^N f\right)^{(m)} \right\|_{\mathbf{u}} \leqslant \frac{c\,(-\lambda_{n-N,0})^{(m-N)/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}} E_{n-r,\mathbf{v}_{r-N-1}}\left(f^{(r)}\right), \quad N \leqslant m \leqslant r,$$

holds always in the Hermite case, for $\alpha > -1$ in the Laguerre case, and for $\alpha, \beta \geqslant 0$ in the Jacobi case.

Proof For $m \geqslant N$, we have

$$\left\| f^{(m)} - \left(\mathcal{S}_n^N f\right)^{(m)} \right\|_{\mathbf{u}} = \left\| \left(f^{(N)} - S_{n-N,0} f^{(N)}\right)^{(m-N)} \right\|_{\mathbf{u}}.$$

Then, the result follows from Remark 2.5 and Theorem 3.7. □

In order to handle the case of derivatives of lower order, we need to introduce the following subspace

$$H_0^N(\mathbf{u}) = \left\{ f \in H^N(\mathbf{u}) : \; f^{(j)}(\nu_i) = 0, \; 0 \leqslant j \leqslant d_i - 1, \; 1 \leqslant i \leqslant s \right\}.$$

Observe that for $f, g \in H_0^N(\mathbf{u})$,

$$(f, g) = \left\langle \mathbf{u}, f^{(N)} g^{(N)} \right\rangle.$$

This means that $\left\langle \mathbf{u}, f^{(N)} g^{(N)} \right\rangle$ is an inner product on $H_0^N(\mathbf{u})$. By the Riesz Representation Theorem, for each $h \in H_0^N(\mathbf{u})$, there is a unique $g \in H_0^N(\mathbf{u})$ such that

$$\langle h\,\mathbf{u}, \; p\rangle = (g, p) = \left\langle \mathbf{u}, \; g^{(N)} p^{(N)} \right\rangle, \quad p \in H_0^N(\mathbf{u}),$$

which is equivalent to

$$\begin{aligned} (-1)^N \left[g^{(N)}\,\mathbf{u}\right]^{(N)} &= h\,\mathbf{u}, \\ g^{(j)}(\nu_i) = 0 \quad 0 \leqslant j &\leqslant d_i - 1, \quad 1 \leqslant i \leqslant s. \end{aligned} \tag{5.1}$$

Moreover, we have the following identity for $v \in H^N(\mathbf{u})$,

$$\left\langle \mathbf{u}, v^{(N)} g^{(N)} \right\rangle = \left\langle (-1)^N \left[g^{(N)}\,\mathbf{u}\right]^{(N)}, v \right\rangle = \langle \mathbf{u}, v\,h \rangle. \tag{5.2}$$

Theorem 5.4 *Let $r \geqslant N+1$ and $f \in H^r(\mathbf{u})$ such that $f^{(r)} \in L^2(\mathbf{v}_{r-N-1})$. Then,*

$$\left\| f^{(m)} - \left(\mathcal{S}_n^N f\right)^{(m)} \right\|_{\mathbf{u}} \leqslant \frac{c\,(-\lambda_{n-N,0})^{N/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}} E_{n-r,\mathbf{v}_{r-N-1}}\left(f^{(r)}\right), \quad 0 \leqslant m \leqslant N-1,$$

always holds in the Hermite case, for $\alpha > -1$ in the Laguerre case, and for $\alpha, \beta \geqslant 0$ in the Jacobi case.

Proof To prove the case $m = 0$, we use the well-known duality argument, the so-called Aubin–Nitsche technique, based on the identity

$$\left\| f - \mathcal{S}_n^N f \right\|_{\mathbf{u}} = \sup_{\|h\|_{\mathbf{u}} \neq 0} \frac{\left\langle \mathbf{u}, h \left(f - \mathcal{S}_n^N f \right) \right\rangle}{\|h\|_{\mathbf{u}}}. \tag{5.3}$$

Let $g \in H_0^N(\mathbf{u})$ be defined as in (5.1). Applying (5.2) with $v = f - \mathcal{S}_n^N f$, we obtain

$$\begin{aligned}\left\langle \mathbf{u}, h \left(f - \mathcal{S}_n^N f \right) \right\rangle &= \left\langle \mathbf{u}, \, g^{(N)} \left(f - \mathcal{S}_n^N f \right)^{(N)} \right\rangle \\ &= \left\langle \mathbf{u}, \, g^{(N)} \left(f^{(N)} - S_{n-N,0} f^{(N)} \right) \right\rangle.\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\left\langle \mathbf{u}, h \left(f - \mathcal{S}_n^N f \right) \right\rangle \leqslant \left\| g^{(N)} \right\|_{\mathbf{u}} \left\| f^{(N)} - S_{n-N,0} f^{(N)} \right\|_{\mathbf{u}},$$

and

$$\left\| g^{(N)} \right\|_{\mathbf{u}}^2 = \langle \mathbf{u}, h\, g \rangle \leqslant \|h\|_{\mathbf{u}} \, \|g\|_{\mathbf{u}}. \tag{5.4}$$

Observe that

$$g^{(N)}(x) = \sum_{j=N}^{\infty} \widehat{g}_{j,0} \, Q_{j,0}^{(N)}(x) = \sum_{j=N}^{\infty} \widehat{g}_{j,0} \, Q_{j-N,N}(x).$$

Using the Parseval identity and (3.4), we get

$$\left\| g^{(N)} \right\|_{\mathbf{u}}^2 = \sum_{j=0}^{\infty} \left((-1)^N \prod_{i=1}^{N} \lambda_{j+i,N-i} \right) \left| \widehat{g}_{j+N,0} \right|^2 h_{j+N,0} \geqslant (-\lambda_{1,N-1})^N \|g\|_{\mathbf{u}}^2.$$

It follows from (5.4) that

$$\|g\|_{\mathbf{u}} \leqslant \frac{1}{(-\lambda_{1,N-1})^N} \|h\|_{\mathbf{u}} \quad \text{and} \quad \left\| g^{(N)} \right\|_{\mathbf{u}} \leqslant \frac{1}{(-\lambda_{1,N-1})^{N/2}} \|h\|_{\mathbf{u}}.$$

Therefore,

$$\frac{\left\langle \mathbf{u}, h \left(f - \mathcal{S}_n^N f \right) \right\rangle}{\|h\|_{\mathbf{u}}} \leqslant \frac{1}{(-\lambda_{1,N-1})^{N/2}} \left\| f^{(N)} - S_{n-N,0} f^{(N)} \right\|_{\mathbf{u}}.$$

By (5.3) and Theorem 5.3, we get

$$\left\| f - \mathcal{S}_n^N f \right\|_{\mathbf{u}} \leqslant \frac{c\,(-\lambda_{n-N,0})^{N/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}} E_{n-r,\mathbf{v}_{r-N-1}}\left(f^{(r)}\right). \tag{5.5}$$

We handle the intermediate case $1 \leqslant m \leqslant N-1$ by following the argument found in [8] (see also [16]): Let T be a function from $[0,\infty)$ to $L^2(\mathbf{u})$:

$$\begin{aligned} T : [0,\infty) &\to L^2(\mathbf{u}) \\ t \quad &\mapsto T(t). \end{aligned}$$

Suppose that T has N continuous derivatives and T and $T^{(N)}$ are bounded on $[0,\infty)$. Then by [8, Theorem I], we have

$$\|T^{(m)}\|_\infty \leqslant c\,\|T\|_\infty^{1-m/N}\,\|T^{(N)}\|_\infty^{m/N}, \quad 1 \leqslant m \leqslant N-1, \tag{5.6}$$

where $\|T\|_\infty = \sup_{t\in[0,\infty)} \|T(t)\|_{\mathbf{u}}$. In particular, consider the *translation operator* A_t defined by $A_t f(x) = f(t+x)$, then A_t can be represented as $A_t = e^{t\frac{d}{dx}}$. Notice that $\|A_t f\|_{\mathbf{u}} \leqslant \|f\|_{\mathbf{u}}$ for all $t \in [0,\infty)$ and $f \in L^2(\mathbf{u})$. Taking into account the above, the set $\{A_t\}_{t\geqslant 0}$ is a contraction semigroup of operators (with infinitesimal generator $\frac{d}{dx}$) on $L^2(\mathbf{u})$. Therefore, for $v \in H^N(\mathbf{u})$, if $T(t) := A_t\, v$, then T is N times continuously differentiable and

$$\|T^{(m)}\|_\infty = \sup_{t\geqslant 0} \|A_t\, v^{(m)}\|_{\mathbf{u}} = \|v^{(m)}\|_{\mathbf{u}}, \quad 0 \leqslant m \leqslant N-1,$$

where we have used

$$T^{(m)}(t) = \frac{d^m}{dt^m} A_t\, v = \frac{d^m}{dt^m} e^{t\frac{d}{dx}}\, v = e^{t\frac{d}{dx}} \frac{d^m v}{dx^m} = A_t\, v^{(m)}.$$

Setting $v = f - \mathcal{S}_n^N f$ and using (5.6), we obtain

$$\left\| f^{(m)} - \left(\mathcal{S}_n^N f\right)^{(m)} \right\|_{\mathbf{u}} \leqslant c\, \left\| f - \mathcal{S}_n^N f \right\|_{\mathbf{u}}^{1-m/N} \left\| f^{(N)} - \left(\mathcal{S}_n^N f\right)^{(N)} \right\|_{\mathbf{u}}^{m/N}.$$

The result follows from Theorem 5.3 and (5.5).

Remark 5.5 Theorems 5.3 and 5.4 with $\widetilde{\mathcal{S}}_n^N$ can be proved with identical proofs.

6 A numerical example

For $\alpha > -1$, let $L_n^\alpha(x)$ denote the n-th monic Laguerre polynomial defined as ([33, Chapter V])

$$L_n^\alpha(x) = (-1)^n\, n! \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{1}{k!} x^k, \quad n \geqslant 0.$$

These polynomials are orthogonal with respect to the positive definite moment functional $\mathbf{u}_\alpha$ defined by

$$\langle \mathbf{u}_\alpha, p(x)\rangle = \int_0^{+\infty} p(x)\, x^\alpha\, e^{-x}\, dx, \quad \forall p \in \Pi.$$

Moreover, we have

$$\left\langle \mathbf{u}_\alpha, L_n^\alpha\, L_m^\alpha \right\rangle = n!\, \Gamma(n+\alpha+1)\, \delta_{n,m}.$$

By Theorem 2.3, the derivatives

$$Q_{n,k}^\alpha(x) \;=\; \frac{d^k}{dx^k} \frac{L_{n+k}^\alpha(x)}{(n+k)!}, \quad n \geqslant 0, \quad k \geqslant 0, \tag{6.1}$$

are orthogonal with respect to $\mathbf{u}_{\alpha+k} \;=\; x^k\, \mathbf{u}_\alpha$; that is,

$$Q_{n,k}^\alpha(x) \;=\; \frac{L_n^{\alpha+k}(x)}{n!}.$$

Consider the numbers $\nu_1 = 0$, $\nu_2 = 1$, $\theta \;=\; 0$, and the polynomial

$$\varphi(x) \;=\; x^2\,(x-1), \quad N := \deg\, \varphi \;=\; 3.$$

The Hermite interpolation polynomial (associated with $\varphi(x)$) for a function f is given by

$$\mathcal{H}_\varphi f(x) \;=\; f(0)\,(1-x^2) + f'(0)\,(x-x^2) + f(1)\,x^2.$$

Then, by Theorem 4.1, there is a 3×3 positive definite symmetric matrix Λ such that the set of polynomials $\{q_n(x)\}_{n\geqslant 0}$ with

$$q_n(x) = \begin{cases} x^n, & 0 \leqslant n \leqslant 2, \\ Q_{n,-3}^{\alpha+3}(x) - \mathcal{H}_\varphi\, Q_{n,-3}^{\alpha+3}(x), & n \geqslant 3, \end{cases} \tag{6.2}$$

where

$$Q_{n,-3}^{\alpha+3}(x) = \frac{1}{2\,(n-3)!} \int_0^x (x-t)^2\, L_{n-3}^{\alpha+3}(t)\, dt, \qquad n \geqslant 3,$$

is orthogonal with respect to the discrete–continuous Sobolev inner product

$$(f, g) = \big(f(0),\, f'(0),\, f(1)\big)\ \Lambda \begin{pmatrix} g(0) \\ g'(0) \\ g(1) \end{pmatrix} + \left\langle \mathbf{u}_{\alpha+3},\, f^{(N)}\, g^{(N)} \right\rangle .$$

Indeed, computing the matrix M in (4.9) and using the fact that $\Lambda = (M^{-1})^{\top} M^{-1}$, we get

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{and} \quad \Lambda = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Using (6.1) and integration by parts, we obtain

$$Q^{\alpha+3}_{n,-3}(x) \;=\; \frac{L^{\alpha}_n(x)}{n!} - \frac{L^{\alpha}_n(0)}{n!} - x\,\frac{L^{\alpha+1}_{n-1}(0)}{(n-1)!} - \frac{x^2}{2}\,\frac{L^{\alpha+2}_{n-2}(0)}{(n-2)!}, \quad n \geqslant 3.$$

Observe that $Q^{\alpha+3}_{n,-3}(0) = (Q^{\alpha+3}_{n,-3})'(0) = (Q^{\alpha+3}_{n,-3})''(0) = 0$, and consequently,

$$\mathcal{H}_{\varphi}\, Q^{\alpha+3}_{n,-3}(x) \;=\; Q^{\alpha+3}_{n,-3}(1)\, x^2, \quad n \geqslant 3.$$

Moreover,

$$\begin{aligned} &(q_n, q_n) = 1, \quad 0 \leqslant n \leqslant 2, \\ &(q_n, q_n) = \left\langle \mathbf{u}_{\alpha+3}, \left(\frac{L^{\alpha+3}_{n-3}}{(n-3)!} \right)^2 \right\rangle = \frac{\Gamma(\alpha+n+1)}{(n-3)!}, \quad n \geqslant 3. \end{aligned}$$

Now, consider the function

$$f(x) \;=\; x^2\, e^{-x}.$$

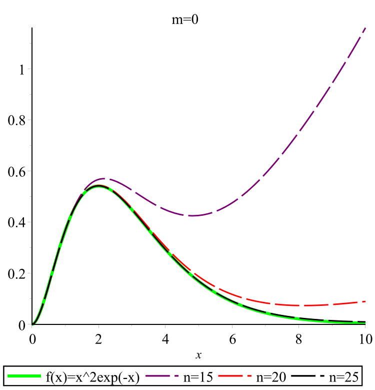
Table 2 Errors of the expansions

r	n	m = 0	m=1	m=2
8	15	0.04873	0.03472	0.031973
	20	0.048483	0.02888	0.036588
	25	0.050662	0.030207	0.042135
	30	0.051321	0.030592	0.045804
	40	0.0500083	0.029841	0.050247
	50	0.047100	0.028092	0.052751
	70	0.041954	0.025049	0.05557
	100	0.036956	0.022062	0.057701
	120	0.034110	0.020373	0.058499

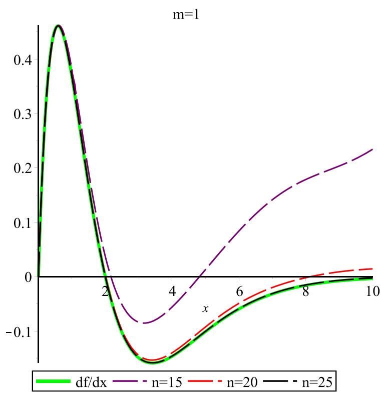
For $\alpha = 0$, Table 2 shows the values of

$$\frac{\left\| f^{(m)} - \left(\mathcal{S}_n^N f\right)^{(m)} \right\|_{\mathbf{u}_\alpha}}{\dfrac{(-\lambda_{n-N,0})^{N/2}}{(-\lambda_{n-r,r-N-1})^{(r-N-1)/2}} E_{n-r,\mathbf{v}_{r-N-1}}\left(f^{(r)}\right)},$$

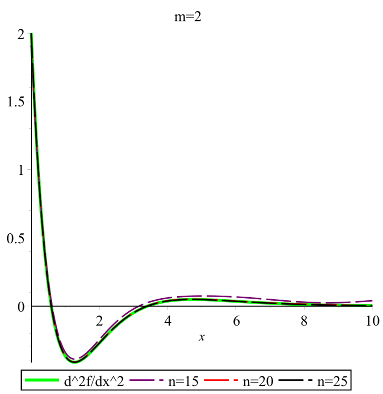
where $\mathcal{S}_n^N$ denotes the projection operator with respect to the Sobolev orthogonal polynomials (6.2). In this case, $\lambda_{n,k} = -n$ (see Table 1) which happens to be independent of k. Since f belongs to C^∞, we choose $r = 8$ and observe that the ratio of the above errors seems to decay as n grows for $m = 0$ and $m = 1$. For $m = 2$, it is likely that the ratio of the errors behaves similarly to the other two cases; however, it seems that it does so much slower.



(A) Approximation of $f(x) = x^2\, e^{-x}$.



(B) Approximation of $f'(x)$.



(C) Approximation of $f''(x)$.

Fig. 1 Simultaneous approximation of f using $\mathcal{S}_n^N f$

Figure 1 depicts the graphs of f, f', and f'' together with the graphs of $\mathcal{S}_n^N f$, $(\mathcal{S}_n^N f)'$, and $(\mathcal{S}_n^N f)''$ for $n = 15, 20, 25$. We note that the projection operator $\mathcal{S}_{25}^N f$ seems to provide an adequately close simultaneous approximation on the interval shown. We should also remark that $f(0) = \mathcal{S}_n^N f(0)$, $f'(0) = (\mathcal{S}_n^N f)'(0)$, and $f(1) = \mathcal{S}_n^N f(1)$ for all values of n.

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Declarations

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References

1. Alfaro, M., Pérez, T.E., Piñar, M.A., Rezola, M.L.: Sobolev orthogonal polynomials: the discrete-continuous case. Methods Appl. Anal. **6**, 593–616 (1999)
2. Alfaro, M., Álvarez de Morales, M., Rezola, M.L.: Orthogonality of the Jacobi polynomials with negative integer parameters. J. Comput. Appl. Math. **145**, 379–386 (2002)
3. Álvarez de Morales, M., Pérez, T.E., Piñar, M.A.: Sobolev orthogonality for the Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n\geqslant 0}$. J. Comput. Appl. Math. **100**, 111–120 (1998)
4. Bochner, S.: Über Sturm-Liouvillesche Polynomsysteme. Math. Z. **29**(1), 730–736 (1929)
5. Bruder, A., Littlejohn, L.L.: Nonclassical Jacobi polynomials and Sobolev orthogonality. Res. Math. **61**, 283–313 (2012)

6. Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral Methods: Fundamentals in Single Domains. Springer, Berlin (2006)
7. Canuto, C., Quarteroni, A.: Approximation results for orthogonal polynomials in Sobolev spaces. Math. Comput. **38**, 67–86 (1982)
8. Certain, M.W., Kurtz, T.G.: Landau-Kolmogorov inequalities for semigroups and groups. Proc. Amer. Math. Soc. **63**, 226–230 (1977)
9. Chihara, T.S.: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)
10. Courant, R., Hilbert, D.: Methods in Mathematical Physics, vol. 1. Wiley-Interscience, New York (1953)
11. Davis, P.J.: Interpolation and Approximation. Dover, New York (1975)
12. Díaz-González, A., Marcellán, F., Pijeira-Cabrera, H., W. Urbina: Discrete-continuous Jacobi-Sobolev spaces and Fourier series. Bull. Malays. Math. Sci. Soc. **44**, no. 2, 571-598 (2021)
13. García-Ardila, J.C., Marcellán, F., Marriaga, M.E.: Orthogonal Polynomials and Linear Functionals. An Algebraic Approach and Applications. European Mathematical Society (EMS), Zürich (2021)
14. García-Ardila, J.C., Marriaga, M.E.: On Sobolev bilinear forms and polynomial solutions of second-order differential equations. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser, A-Mat (2021)
15. García-Caballero, E.M., Pérez, T.E., Piñar, M.A.: Sobolev orthogonal polynomials: interpolation and approximation. Electron. Trans. Numer. Anal. **9**, 56–64 (1999)
16. Gesztesy, F., Nichols, R., Stanfill, J.A.: Survey of Some Norm Inequalities. Complex Anal. Oper. Theory **15**, no. 2, Paper No. 23, 33 pp (2021)
17. García-Caballero, E.M., Pérez, T.E., Piñar, M.A.: Hermite interpolation and Sobolev orthogonality. Acta Appl. Math. **61**:87–99 (2000)
18. Guo, B.Y., Shen, J., Wang, L.L.: Optimal spectral-Galerkin methods using generalized Jacobi polynomials. J. Sci. Comput. **27**, 305–322 (2006)
19. Guo, B.Y., Shen, J., Wang, L.L.: Generalized Jacobi polynomials/functions and applications to spectral methods. Appl. Numer. Math. **59**, 1011–1028 (2009)
20. Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Presss, Cambridge (2013)
21. Koekoek, R., Meijer, H.G.: A generalization of Laguerre polynomials. SIAM J. Math. Anal **24**, 768–782 (1993)
22. Krall, H.L.: On derivatives of orthogonal polynomials II. Bull. Amer. Math. Soc. **47**, 261–264 (1941)
23. Kwon, K.H., Littlejohn, L.L.: The orthogonality of the Laguerre polynomials $\{L_n^{(-k)}(x)\}$ for positive integers k. Ann. Numer. Anal. **2**, 289–304 (1995)
24. Li, H., Xu, Y.: Spectral approximation on the unit ball. SIAM J. Numer. Anal. **52**, 2647–2675 (2014)
25. Magomed-Kasumov, M.G.: Sobolev Orthogonal Systems with Two Discrete Points and Fourier Series, Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2021, No. 12, pp. 56-66. translation in Russian Mathematics, 2021, **65**, No. 12, pp. 47-55 (2021)
26. Marcellán, F., Branquinho, A., Petronilho, J.: Classical orthogonal polynomials: a functional approach. Acta Appl. Math. **34**, 283–303 (1994)
27. Marcellán, F., Petronilho, J.: On the solution of some distributional differential equations: existence and characterizations of the classical moment functionals. Integral Transform. Spec. Funct. **2**(3), 185–218 (1994)
28. Marcellán, F., Xu, Y.: On Sobolev orthogonal polynomials. Expo. Math. **33**(3), 308–352 (2015)
29. Sánchez-Lara, J.F.: On the Sobolev orthogonality of classical orthogonal polynomials for non standard parameters. Rocky Mountain J. Math. **47**(1), 267–288 (2017)
30. Sharapudinov, I.I.: Approximation properties of Fourier series of Sobolev orthogonal polynomials with Jacobi weight and discrete masses. Mat. Zametki 101 (2017), no. 4, 611-629; translation in Math. Notes **101**, no. 3-4, 718-734 (2017)
31. Sharapudinov, I.I.: Sobolev-orthogonal systems of functions and some of their applications. Uspekhi Mat. Nauk **74** (2019), no. 4(448), 87-164; translation in Russian Math. Surveys **74**, no. 4, 659-733 (2019)
32. Xu, Y.: Approximation and orthogonality in Sobolev spaces on a triangle. Constr. Approx. **46**, 349–434 (2017)
33. Szegő, G.: Orthogonal polynomials, 4th edition, vol. 23. Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., Providence RI (1975)

34. Xu, Y.: Approximation by polynomials in Sobolev spaces with Jacobi weight. J. Fourier Anal. Appl. **24**, 1438–1459 (2018)
35. Yu, X., Wang, Z., Li, H.: Jacobi-Sobolev Orthogonal Polynomials and Spectral Methods for Elliptic Boundary Value Problems, Commun. Appl. Math. Comput. **1**, 283–308 (2019)