We deal with the relation between the Mittag-Leffler functions and the sum of the hyper-volume of generalized l^p n-balls. We derive from this result some simple properties of the Mittag-Leffler functions. We also define some new Abelian groups from the sum and the multiplication as well as the Pontryagin transform associated to them. We relate this approach to the calculus on measure chains.

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Mittag-Leffler functions, l^p n-balls and geometric interpretations

Gorka Guardiola Múzquiz[a*]

[a] *Rey Juan Carlos University, Madrid, Spain*

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1. Introduction

Along the paper, $\mathbb{N} = \{1, 2, 3 \ldots\}$ will denote the natural numbers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the integer numbers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. $\mathbb{R}^+$ are the strictly positive real numbers $x > 0$.

The special function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \ \alpha \in \mathbb{R}, z \in \mathbb{C}, \tag{1.1}$$

is called Mittag-Leffler function [1] [2]. It arises naturally as the eigenfunction of the iterated derivative operator, generalizing the exponential function:

$$D^k E_k(\gamma z^k) = \gamma E_k(\gamma z^k), k \in \mathbb{N}, \gamma \in \mathbb{R}. \tag{1.2}$$

(1.2) can be extended to real indexes by using the different operators of fractional calculus [3]. For example, let D^α_{0+}, $\alpha > 0$ be the *right-sided operator of Riemann-Liouville fractional derivative*,

$$(D^\alpha_{0+} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \left(I^{1-\{\alpha\}}_{0+} f\right)(x), \tag{1.3}$$

where $[\alpha]$ means the maximal integer number not exceeding α, $\{\alpha\}$ is the fractional part of α and I^α_{0+} is the *right-sided operator of the Riemann-Liouville fractional integral*,

$$\left(I^\alpha_{0+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt. \tag{1.4}$$

*Corresponding author. Email: paurea@gmail.com

From [3],

$$\left(D_{0+}^{\alpha}[E_{\alpha}(\gamma t^{\alpha})]\right)(z) = \frac{z^{-\alpha}}{\Gamma(1-\alpha)} + \gamma E_{\alpha}(\gamma z^{\alpha}),\ \alpha > 0,\ a \in \mathbb{R}. \tag{1.5}$$

Notice that the Mittag-Leffler functions are eigenfunctions of this operator only when α is an integer and the first term, $\frac{z^{-\alpha}}{\Gamma(1-\alpha)}$, cancels.

A generalized version of the Mittag-Leffler function of two indexes, known as Wiman function, is defined [4] as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)},\ \ \alpha, \beta \in \mathbb{R}, z \in \mathbb{C}. \tag{1.6}$$

It can be generalized further, see [5], [2] and [6]. One of such generalizations are the generalized γ-hyperbolic functions

$$F_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\gamma^k z^{\beta+\alpha k}}{\Gamma(\beta+\alpha k+1)} = z^{\beta} E_{\alpha,\beta+1}(\gamma z^{\alpha}),\ \alpha, \beta \in \mathbb{R},\ \gamma, z \in \mathbb{C}, \tag{1.7}$$

which satisfy [6] a different version of (1.2)

$$D^k F_{k,n}^{\gamma}(z) = \gamma F_{k,n}^{\gamma}(z),\ k, n \in \mathbb{N},\ k \geq n. \tag{1.8}$$

Note that in the literature they are known as α-hyperbolic functions, but we have changed α to γ in order to unify the notation with Mittag-Leffler functions. (1.8) is a special case of Th. 5 in [3], which generalizes (1.5)

$$\left(D_{0+}^{\alpha}[F_{\alpha,\beta}^{\gamma}(t)]\right)(z) = \frac{z^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} + \gamma F_{\alpha,\beta}^{\gamma}(z),\ \alpha > 0,\ \beta > 0. \tag{1.9}$$

Note that, when $\beta - \alpha \in \mathbb{Z}$, the first term is zero.

In 1757, V. Riccati recorded the first generalizations of hyperbolic functions, assuming α and β are integer numbers and $\alpha \geq 2$ and $0 \leq \beta < \alpha$, but there is no reason for such restrictions in general. On the other hand, many modern treatments considering non-integer indexes like [3] fail to relate the properties of both families, generalized Mittag-Leffler and γ-hyperbolic functions.

Section 2 deals with the series for the Mittag-Leffler family of functions from a novel geometric perspective setting up the basic background needed for the rest of the paper. Section 3 defines some LCA (locally σ-compact Abelian groups) groups in some sense between the sum and multiplication and based on Mittag-Leffler functions. In section 4 we consider the Pontryagin duality transforms associated with these groups. These transforms yield a parametrized interval spanning from the Fourier transform to a variation of the Mellin transform. In section 6 we consider some other Fourier transforms and Lie groups obtained by other authors using a different approach but related to the same family of functions. In section 7 we relate our results with the ones obtained with the calculus on time scales. Section 8 generalizes the group definitions of previous sections to an infinite family of groups. Finally, conclusions and future work are presented in section 9.

2. Basic background

Mittag-Leffler functions are related to the hypervolume of a generalized l^p n-ball. This is the ball B_n^p of constant p-norm i.e. the set of elements of l^p with $p > 1$ such that $\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}} \leq R$. The volume of B_n^p, as calculated by Wang and previously by Dirichlet and others is (see the history of this result in the paper by Wang [8]):

$$V_n^p(R) = \frac{(2\Gamma(\frac{1}{p}+1)R)^n}{\Gamma(\frac{n}{p}+1)}. \tag{2.1}$$

Comparing (2.1) and (1.1), when $p = \frac{1}{\alpha}$ we get

$$E_\alpha(2\Gamma(\alpha+1)R) = \sum_{n=0}^{\infty} \frac{(2\Gamma(\alpha+1)R)^n}{\Gamma(\alpha n+1)} = \sum_{n=0}^{\infty} V_n^{\frac{1}{\alpha}}(R). \tag{2.2}$$

A different way to state this is that, if $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, then the Mittag-Leffler function is the sum of the volumes of generalized n-balls with $\alpha = \frac{1}{p}$ and radius $R = \frac{t}{2\Gamma(\alpha+1)}$.

If $R \neq 0$ and $\alpha \neq 0$, (2.2) can be written in terms of generalized $\gamma - hyperbolic$ functions, $E_\alpha(2\Gamma(\alpha+1)R) = \frac{F_{\alpha,0}^{\Gamma(\alpha+1)}((2R)^p)}{(2R)^p}$.

In fact, the consequences of the result by Wang are stronger; the volume for hyper-ellipsoids is related to further generalizations of the Mittag-Leffler function.

Also, the restriction $0 \leq \alpha \leq 1$ can be lifted, and then the result is not a norm. We do not consider this case here.

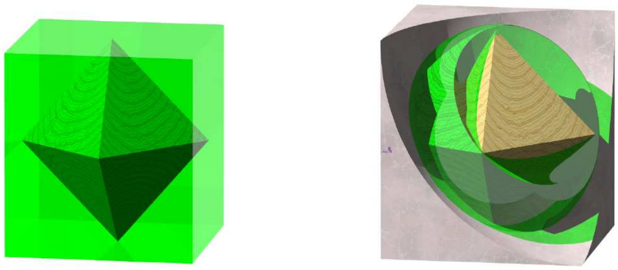
The generalized version of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is related to the volumes of what we can call generalized n-pseudocylinders, n-balls which have as sections in r-dimensional balls of $\frac{1}{p} = \alpha$ and in q dimensions $\frac{1}{s} = \beta$. The volume W of these n-pseudocylinders is:

$$W_n^{p,s}(R) = \frac{\Gamma(\frac{1}{p}+1)^r \Gamma(\frac{1}{s}+1)^q (2R)^n}{\Gamma(\frac{r}{p}+\frac{s}{q}+1)}. \tag{2.3}$$

Thus,

$$\begin{aligned}\sum_{r=0}^{\infty} W_n^{p,s}(R) &= \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{p}+1)^r \Gamma(\frac{1}{s}+1)^q (2R)^n}{\Gamma(\frac{r}{p}+\frac{q}{s}+1)} = \Gamma\Big(\frac{1}{s}+1\Big)^q \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{p}+1)^r (2R)^n}{\Gamma(\frac{r}{p}+\frac{q}{s}+1)} \\ &= (\Gamma(\beta+1)2R)^q \sum_{r=0}^{\infty} \frac{(\Gamma(\alpha+1)2R)^r}{\Gamma(r\alpha+q\beta+1)} = (\Gamma(\beta+1)2R)^q E_{\alpha,1+q\beta}(\Gamma(\alpha+1)2R) \\ &= \Gamma(\beta+1)^q (2R)^{\frac{q(\alpha-\beta)}{\alpha}} F_{\alpha,q\beta}^{\Gamma(\alpha+1)}((2R)^{\frac{1}{\alpha}}),\ \alpha \neq 0.\end{aligned}$$

We believe that these geometric relations have not been pointed out before.



(a) Cube and octahedron. (b) Cube, sphere and octahedron.

Figure 1. Example of a polytype, cross-polytype and sphere in $\mathbb{R}^3$.

Instead of defining the volume of the ball with l_p norm $\|x\|_p = (\sum_i |x_i|^p)^{\frac{1}{p}} \leq R$ we could use the slightly different definition for the volume, defined as the space delimited by $\|x\|_p^p = \sum_i |x_i|^p \leq R$. Looking at (1.5) this is, in some sense, a more natural definition (we use here H to distinguish from the volume V defined above):

$$E_\alpha(2\Gamma(\alpha+1)z^\alpha) = \sum_{n=0}^{\infty} H_n^{\frac{1}{\alpha}}(z). \tag{2.4}$$

While equivalent, this definition yields two problems. First the introduction of complex roots for negative numbers (or multiple roots even for real numbers). Second, the limit for $\alpha = 0$ would yield a constant volume independent of the radius and hence no insight.

(2.2) has several consequences. First, the result by Pollard [9] is, of course, trivial for $R > 0$. Pollard result means that in the region $1 < \alpha \leq \infty$ (with z restricted to $\mathbb{R}$) the function is completely monotonic. When $R > 0$ the volume grows with the radius. This result can be made stronger for $R > 0$ by realizing that the area of the generalized n-balls is proportional to the derivative of the volume (by the co-area formula). Thus, the derivative is not only positive but also increases monotonically.

Looking at the n-balls, at one end, $\alpha = 0$, we get the sum of all cross-polytopes (orthoplexes) of dimension n, at the other $\alpha = 1$ the hypercubes of dimension n. In the middle, the sum of all hyperspheres. For example, if $n = 3$, we get a regular octahedron, a cube and a 3-sphere, as is shown in Figures 1(a) and 1(b).

The problem stated by Freden [10], i.e. the sum of all hyperspheres of the same radius is a special case of the Mittag-Leffler function from [1]:

$$E_{\frac{1}{2}}(z) = e^{z^2}(1 + erf(z)) = erfcx(-z) = F^1_{\frac{1}{2},0}(z^2). \tag{2.5}$$

Applying (2.5), as well as $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ the volume obtained by Freden is:

$$\sum_{n=0}^{\infty} V_n^2(R) = erfcx\left(-2R\frac{\sqrt{\pi}}{2}\right) = erfcx(-\sqrt{\pi}R) = F^{\sqrt{\pi}}_{\frac{1}{2},0}(R^2). \tag{2.6}$$

Another special case is the exponential $E_1(z) = e^z$ which makes $E_1(2\Gamma(2)R) = e^{2\Gamma(2)R} = e^{2R} = F^2_{1,0}(R)$

the sum of the hypervolume of all the hyperdiamonds, i.e. the balls of the standard infinite norm $\|x\|_\infty = max_i\{|x_i|\}$. The edge of a cross-polytope is proportional to the radius $x = R\sqrt{2}$ [11], so the added volume of all cross-polytopes of constant edge is $\sum_{n=0}^{\infty} \frac{(x/\sqrt{2})^n}{n!} = e^{x/\sqrt{2}} = F_{1,0}^{\frac{1}{\sqrt{2}}}(x)$.

The function in $\alpha = 0$ is, for $|z| < 1$, $E_0(z) = \frac{1}{1-z}$. So the sum of the hypervolume of all the hypercubes, i.e. the balls of the l_1 norm $\|x\|_1 = \sum_{i=0}^{n} |x_i|$, is $E_0(2\Gamma(1)R) = E_0(2R) = \frac{1}{1-2R}$.

The case of the exponential function is remarkable, because the even part of the Mittag-Leffler of $\alpha = \frac{1}{2}$, applying (2.5) is $even\Big(E_{\frac{1}{2}}(z)\Big) = \frac{E_{\frac{1}{2}}(z)+E_{\frac{1}{2}}(-z)}{2} = e^{z^2} = F_{1,0}^1(z^2)$ which gives us the result that the sum of the volumes of hyperspheres of even dimension $d = 2n$ is $\sum_{n=0}^{\infty} V_{2n}^2(R) = e^{\pi R^2}$. This is a special case of the representation of the Mittag-Leffler function from [1]: $E_\alpha(z) = \frac{1}{m} \sum_{r=0}^{m-1} E_{\frac{\alpha}{m}}(z^{\frac{1}{m}} e^{\frac{i2\pi r}{m}}),\ m \in \mathbb{N}$.

When $m = 2$ and $\alpha = 1$, $E_1(z) = \frac{1}{2}E_{\frac{1}{2}}(z^{\frac{1}{2}}) + \frac{1}{2}E_{\frac{1}{2}}(-z^{\frac{1}{2}}) = e^z$ and, as a consequence, if $z = R^2$, we get $E_1(R^2) = e^{R^2}$.

3. Mittag-Leffler induced groups

For $0 \le \alpha \le 1$, let us define implicitly the family of operations:

$$x \circledast_\alpha y = E_\alpha^{-1}(E_\alpha(x)E_\alpha(y)), \tag{3.1}$$

where x and y are real numbers and E_α^{-1} is the inverse of the Mittag-Leffler function. When $0 < \alpha \le 1$, the inverse function is defined because E_α is holomorphic and monotonic [9]. When $\alpha = 0$, then $E_0(x) = \frac{1}{1-x}$ and some care is needed when $x = 1$, where the function is not continuous. When $x \neq 1$, the function has a simple inverse:

$$E_0^{-1}(x) = \frac{x-1}{x}. \tag{3.2}$$

Theorem 3.1 *($\circledast_\alpha,\ 0 \le \alpha \le 1,\ \mathbb{R}$) is an Abelian group.*

Proof. • Commutativity and Associativity come from (3.1). Indeed,

$$E_\alpha^{-1}(E_\alpha(x)E_\alpha(y)) = E_\alpha^{-1}(E_\alpha(y)E_\alpha(x))$$

and for $E_\alpha^{-1}(E_\alpha(E_\alpha^{-1}(E_\alpha(x)E_\alpha(y)))E_\alpha(z)) = E_\alpha^{-1}(E_\alpha(x)E_\alpha(E_\alpha^{-1}(E_\alpha(y)E_\alpha(z))))$, we can cancel the inverses and each side equals $E_\alpha^{-1}(E_\alpha(x)E_\alpha(y)E_\alpha(z))$.

- Identity element: $x \circledast_\alpha 0 = x$ because $E_\alpha(0) = 1$, so $x \circledast_\alpha 0 = E_\alpha^{-1}(E_\alpha(x)E_\alpha(0)) = E_\alpha^{-1}(E_\alpha(x)) = x$
- Inverse element y: $x \circledast_\alpha y = 0$, then $E_\alpha^{-1}(E_\alpha(x)E_\alpha(y)) = 0$. So, $E_\alpha(x)E_\alpha(y) = 1$. Then $y = inv_\alpha(x) = E_\alpha^{-1}(\frac{1}{E_\alpha(x)})$. $inv_\alpha(x)$ can be defined because if $x \in \mathbb{R}$ and $0 \le \alpha \le 1$, $E_\alpha(x)$ does not vanish [9]. For E_0, with $x \neq 1$ the inverse is $inv_0(x) = \frac{x}{x-1}$.

- Closure: the function E_α defined in (1.1) is holomorphic on $\mathbb{C}$ up to $\alpha = 0$, that must be excluded. As a consequence, E_α is defined for all $\mathbb{R}$. Furthermore if $x \in \mathbb{R}$ then $E_\alpha(x) \in \mathbb{R}$ and $E_\alpha^{-1}(x) \in \mathbb{R}$.
  When $\alpha = 0$, the series defined in (1.1) only converges for $|z| < 1$ and can be written [1] $E_0(z) = \frac{1}{1-z}$. Applying (3.1) and (3.2) for the inverse, we obtain: $x \circledast_0 y = x + y - xy$. For $x = -1$ and $y = -1$, $-1 \circledast_0 -1 = -3$ which falls out of the radius of convergence. The function E_0 can be analytically extended to $z > 1$. As $1 \circledast_0 1 = 1$, from $x + y - xy = 1$ and $y \neq 1$ you get $x = \frac{1-y}{1-y} = 1$, so the operation $\circledast_0$ is closed for $\mathbb{C} \setminus \{1\}$ or $\mathbb{R} \setminus \{1\}$.

■

Notice that $\circledast_0$ can also be written $x \circledast_0 y - 1 = -(x-1)(y-1)$. It is just a reflected multiplication with a change of origin. On the other hand, $\circledast_1$ is a regular addition.

Lets introduce $\Rsh_\alpha$, the operation generated by iterated composition, i.e. by:

$$\underbrace{x \circledast_\alpha x \circledast_\alpha \cdots \circledast_\alpha x}_{n} = E_\alpha^{-1}(E_\alpha(x)^n) = x \Rsh_\alpha n. \tag{3.3}$$

We have defined the operation by induction, but the above formula defines two cases which implicitly constitute the base for the induction ($n = 0$ and $n = 1$). These are defined when $\alpha = 1$ and the induced operation $\Rsh_1$ is the product:

$$x \Rsh_\alpha 0 = E_\alpha^{-1}(E_\alpha(x)^0) = E_\alpha^{-1}(1) = 0, x \Rsh_\alpha 1 = E_\alpha^{-1}(E_\alpha(x)^1) = x. \tag{3.4}$$

Another case is $0 \Rsh_\alpha y = E_\alpha^{-1}(E_\alpha(0)^y) = E_\alpha^{-1}(1^y) = 0$.

Notice that, in general, the above operation is non-commutative, taking into account the definition is asymmetric.

For $y \in \mathbb{R}$ we would like to define $x \Rsh_\alpha y$. In order to prove (3.3) for $n \in \mathbb{R}$, we proceed in a similar way to the procedure defined to extend the exponential function described in [25]. For $\alpha = 0$, the operation can be obtained explicitly from the definitions, $x \Rsh_\alpha y = 1 - (1-x)^y$, and we have nothing to prove.

As stated for $0 < \alpha \leq 1$ and $x \in \mathbb{R}$, $E_\alpha(x)$ is holomorphic, monotonic and does not vanish. As a consequence, its inverse, $E_\alpha^{-1}(x)$ is also holomorphic and monotonic (but has a zero at $E_\alpha^{-1}(1)$). The function x^n is increasing for $x > 0$ and $E_\alpha(x) > 0$ for $x \in \mathbb{R}$. $E_\alpha^{-1}(E_\alpha(x)^n)$ is continuous, strictly increasing and takes arbitrarily large values, because it is the composition of continuous and strictly increasing functions. Therefore by Bolzano's intermediate value theorem any positive number y has a unique n^{th} $\Rsh_\alpha$ root, which we write $y \Rsh_\alpha \frac{1}{n}$, i.e. $y = E_\alpha^{-1}(E_\alpha(x)^n)$. Solving for x, the explicit expression is

$$x = E_\alpha^{-1}(E_\alpha(y)^{\frac{1}{n}}).$$

Thus, we can define rational powers of a number by setting:

$$x \Rsh_\alpha \frac{m}{n} = (x \Rsh_\alpha \frac{1}{n}) \Rsh_\alpha m.$$

It is easy to check that all the rules of exponents run for rational exponents. For example:

$$(x \Rsh_\alpha \frac{1}{n}) \Rsh_\alpha n = E_\alpha^{-1}(E_\alpha(E_\alpha^{-1}(E_\alpha(x)^{\frac{1}{n}}))^n) = E_\alpha^{-1}((E_\alpha(x)^{\frac{1}{n}})^n) = E_\alpha^{-1}((E_\alpha(x)) = x.$$

When $\gamma \in \mathbb{R}$ is not a rational number, we can define a sequence $\gamma_1, \gamma_2, \gamma_3, ...$ of rational numbers which converges to γ. $y = E_\alpha^{-1}(E_\alpha(x)^\gamma)$ is a continuous function on γ, so it preserves limits and $\mathbb{R}^+$ is complete, so we define $x \Rsh_\alpha \gamma$ to be the limit of the sequence $x \Rsh_\alpha \gamma_i$. We can also define negative exponents by using the inverse element of the group:

$$\begin{aligned} x \Rsh_\alpha 0 = 1 = x \Rsh_\alpha (1-1) &= E_\alpha^{-1}(E_\alpha(x)^{1-1}), \\ &= E_\alpha^{-1}(E_\alpha(x)^1 E_\alpha(x)^{-1}), = (x \Rsh_\alpha 1) \circledast_\alpha (x \Rsh_\alpha -1), \\ &= x \circledast_\alpha (x \Rsh_\alpha -1). \end{aligned}$$

So $E_\alpha^{-1}(\frac{1}{E_\alpha(x)}) = x \Rsh_\alpha -1$. It is easy to check that all the familiar rules for the exponential continue to work for this case also.

4. Pontryagin duality

The group defined above $(\circledast_\alpha,\ \mathbb{R})$ is a topological group with the usual topology, because the operation map $x \circledast_\alpha y = E_\alpha^{-1}(E_\alpha(x)E_\alpha(y))$ and the inverse map $E_\alpha^{-1}(\frac{1}{E_\alpha(x)})$ are both continuous. It is also locally compact, σ-compact and Hausdorff, since the usual topology of $\mathbb{R}$ is.

For LCA, locally σ-compact groups (local, σ-compact, Abelian, Hausdorff) we can define a translation invariant measure (Haar measure). To prove the existence of a Haar measure on a locally σ-compact group G we must exhibit a right-invariant (or in general translation-invariant in our case, because the groups are Abelian) Radon measure on G.

For an LCA locally σ-compact measure space, the definition of a Radon measure can be given in terms of continuous linear functionals on the space of continuous functions with compact support. The Riesz representation theorem associates to every linear non-negative functional a unique Radon measure. If we prove it is translation-invariant, by the Haar theorem, it has a Haar measure, which is the only translation-invariant measure up to a scalar.

Theorem 4.1 *Given S, a compact subset of $(\circledast_\alpha,\ \mathbb{R})$, its measure is defined by:*

$$\mu(S) = \int_S \frac{E'_\alpha(x)}{E_\alpha(x)} dx \tag{4.1}$$

which is translation invariant by the group operation $(\circledast_\alpha,\ \mathbb{R})$.

Proof. First, the linear functional $I(f)$:

$$I(f) = \int_G f(x) \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx = \int_{-\infty}^{\infty} f(x) \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx \tag{4.2}$$

is positive because both $E_\alpha(x)$ and $E'_\alpha(x)$ are strictly positive when $0 < \alpha \leq 1$. When $\alpha = 0$ we have to change the definition of the functional (and hence the measure) slightly, but the rest of the proof is similar

$$I(f) = \int_{-\infty}^{1} f(x) \frac{E'_0(x)}{E_0(x)}\, dx - \int_{1}^{\infty} f(x) \frac{E'_0(x)}{E_0(x)}\, dx. \tag{4.3}$$

We only need to prove the measure of a compact subset S, is right-translation invariant, but

$$\begin{aligned}
\mu(S) &= \int_{S \circledast_\alpha a} \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx, \text{by compacity,} \\
&= \sum_n \int_{x_n \circledast_\alpha a}^{y_n \circledast_\alpha a} \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx = \sum_n ln(E_\alpha(x_n \circledast_\alpha a)) - ln(E_\alpha(y_n \circledast_\alpha a)) \\
&= \sum_n ln(E_\alpha(x_n)E_\alpha(a)) - ln(E_\alpha(y_n)E_\alpha(a)) = \sum_n ln(E_\alpha(x_n)) - ln(E_\alpha(y_n)) \\
&= \sum_n \int_{x_n}^{y_n} \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx = \int_S \frac{E'_\alpha(x)}{E_\alpha(x)}\, dx.
\end{aligned}$$

■

A multiplicative character $\chi(x)$ is a continuous function from the group to the unit circle which is an homomorphism, i.e. $\chi_\alpha(x \circledast_\alpha y) = \chi_\alpha(x)\chi_\alpha(y)$. We can define such a character as

$$\chi_\alpha(x) = e^{\beta 2\pi i ln(E_\alpha(x))} = E_\alpha(x)^{\beta 2\pi i}. \tag{4.4}$$

THEOREM 4.2 *All characters of this group are of the form* (4.4).

Proof. All the characters of the additive group of R are of that form (see [26], Th. 35C). The function E_α establishes a one-to-one correspondence by the (4.4) from the characters of the additive group to the characters of ($\circledast_\alpha$, $\mathbb{R}$). The existence of other characters would imply the same on the additive group, which is barred by [26], Th. 35C. ■

By Pontryagin duality we can use the characters to define a Fourier transform for L^1 functions of compact support for $0 < \alpha \leq 1$:

$$\hat{f}(\beta) = \int_G f(x)\overline{\chi_\alpha(x)}\, d\mu(x) = \int_{-\infty}^{\infty} f(x) E_\alpha(x)^{-\beta 2\pi i - 1} E'_\alpha(x)\, dx. \tag{4.5}$$

We can handle (7.4) to obtain some alternative expressions:

$$\hat{f}(\beta) = \begin{cases} \frac{1}{(-\beta 2\pi i)} \int_{-\infty}^{\infty} f(x) \frac{d}{dx} E_\alpha(x)^{-\beta 2\pi i}\, dx, & \beta \neq 0, \\ \int_{-\infty}^{\infty} f(x)\, d\mu(x), & \beta = 0. \end{cases} \tag{4.6}$$

Integrating by parts, and using the fact that $f(x)$ is integrable and so converges to 0 in $\pm\infty$ and from $E_{\alpha,\alpha}(x) = \alpha \frac{dE_\alpha(x)}{dx}$, [27] so:

$$\hat{f}(\beta) = \int_{-\infty}^{\infty} f(x) \frac{E_\alpha(x)^{-\beta 2\pi i - 1} E_{\alpha,\alpha}(x)}{\alpha}\, dx. \tag{4.7}$$

For $\alpha = 1$, $E_1(x) = e^x$, and we recover the regular Fourier transform.

$$\hat{f}(\beta) = \int_G f(x) \frac{e^{-x\beta 2\pi i} e^x}{e^x} \, dx = \int_{-\infty}^{\infty} f(x) e^{-x\beta 2\pi i} \, dx. \tag{4.8}$$

For $\alpha = 0$, $E_0(x) = \frac{1}{1-x}$, using the functional defined in (4.3) we obtain a variant of the Mellin transform

$$\hat{f}(\beta) = \int_{-\infty}^{1} f(x)(1-x)^{\beta 2\pi i - 1} \, dx - \int_{1}^{\infty} f(x)(1-x)^{\beta 2\pi i - 1} \, dx. \tag{4.9}$$

For $\alpha = \frac{1}{2}$, $E_{\frac{1}{2}} = erfcx(-x)$ and $E'_{\frac{1}{2}} = 2x \; erfcx(-x) + \frac{2}{\sqrt{\pi}}$:

$$\hat{f}(\beta) = \int_{-\infty}^{\infty} 2f(x) \; erfcx(-x)^{-2\pi\beta i - 1} \left(x \; erfcx(-x) + \frac{1}{\sqrt{\pi}} \right) dx. \tag{4.10}$$

We can define the convolution of two functions in our space as:

$$(f *_\alpha g)(x) = \int_G f(y) g((y \Rsh_\alpha -1)x) \, d\mu_\alpha(y). \tag{4.11}$$

From Pontryagin $(\widehat{f *_\alpha g})(x) = \widehat{f(x)}\widehat{g(x)}$.

Applying Pontryagin duality, $\hat{G}$ is also an LCA group, so it has its own Haar measure and further on, its dual is G. We can use this fact to obtain inverse formulas for our generalized transforms:

$$f(x) = \int_{\hat{G}} \hat{f}(\beta) \chi_\alpha(\beta) \, d\nu(\beta). \tag{4.12}$$

Here the equality $f = \hat{\hat{f}}$ is understood to be in terms of the measure, i.e. everywhere up to a set of μ-measure zero. The measure ν is the dual of the measure μ. The group G and its dual $\hat{G}$ can be identified via the characters and the measure ν and its dual μ are the same measure but for a constant C_α. With our choice of sign and constant in the exponent, both can be shown to be the same for $\alpha = 1$, i.e. Fourier transforms (with $C_1 = 1$).

$$f(x) = \int_{\hat{G}} \hat{f}(\beta) \chi_\alpha(\beta) C_\alpha \, d\mu(\beta). \tag{4.13}$$

The group and all the arguments above can be extended to a group obtained from the function (1.6), $x \circledast_{\alpha,\beta} y = E^{-1}_{\alpha,\beta}(E_{\alpha,\beta}(x) E_{\alpha,\beta}(y))$, with the special case when $\beta \neq 0$ $x \circledast_{0,\beta} y = 1 - \Gamma(\beta) + \Gamma(\beta)(x + y - xy)$. Since $E_{\alpha,\beta}(z)$ is continuous (when $\alpha \neq 0$, where the function is $\frac{E_0(z)}{\Gamma(\beta)}$, so all the arguments for that case apply) and when $\beta \geq \alpha$ it is completely monotonic [27]. Note that there is a new (degenerate) case when $\beta = 0$ and $\alpha = 0$: $E_{0,0}(x) = 0$. Other than that, there are no zeroes. From Lemma 2 in [27]:

$$E_{\alpha,\beta}(-x) = \frac{1}{\alpha \Gamma(\beta - \alpha)} \int_0^1 (1 - t^{\frac{1}{\alpha}})^{\beta - \alpha - 1} E_{\alpha,\alpha}(-tx) \, dt, \tag{4.14}$$

and the function inside the integral is nonnegative, $1 - t^{\frac{1}{\alpha}} \geq 0$ when $0 \leq t \leq 1$. But $E_{\alpha,\alpha}(x) > 0$ because it is the derivative of $E_\alpha(x)$ (also in [27]), so the integral is always positive.

The only question that remains to be answered is the existence of an identity for the group, which requires $E_{\alpha,\beta}(x_{id}) = 1$. The value at zero, $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}$ and when $0 \leq \beta \leq 1$, $\Gamma(\beta) \geq 1$, so $E_{\alpha,\beta}(0) < 1$. When $\alpha \neq 0$, $E_{\alpha,\beta}(x)$ is continuous, and can return arbitrarily high values, so by Bolzano' s intermediate value theorem, x_{id} exists. When $\alpha = 0$, $E_{0,\beta}(x_{id}) = \frac{1}{\Gamma(\beta)(1-x_{id})} = 1$, so x_{id} can be given explicitly as $x_{id} = \frac{\Gamma(\beta)-1}{\Gamma(\beta)}$.

Note the special case $E_{1,2}(x) = \frac{e^x-1}{x}$. The identity is $x_{id} = 0$.

5. Group starting with the addition

Another direction to explore is the study of the properties for the operations defined with respect to the sum instead of the multiplication.

For $x, y, z \in R^+$ and $\alpha \geq \beta$

$$z = x \oplus_{\alpha,\beta} y = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x) + E^{-1}_{\alpha,\beta}(y)). \tag{5.1}$$

As stated before, the function $E_{\alpha,\beta}$ is continuous (when $\alpha \neq 0$, where the function is just a constant times E_0, so all the arguments for that case apply) and when $\beta \geq \alpha$ it is completely monotonic [27].

THEOREM 5.1 (($z = x \oplus_{\alpha,\beta} y,\ \alpha \leq \beta,\ 0 \leq \alpha \leq 1,\ x, y, z > 0$) is an Abelian group) *The proof is similar to the one for* $\circledast$.

Proof. The following properties hold:

- Commutativity and associativity come from the addition.
- Identity element $x \oplus_{\alpha,\beta} y_{id} = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x) + E^{-1}_{\alpha,\beta}(y_{id})) = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x)) = x$, so $E^{-1}_{\alpha,\beta}(y_{id}) = 0$, which means $y_{id} = E_{\alpha,\beta}(0) = \frac{1}{\beta}$.
- Inverse element $x \oplus_{\alpha,\beta} inv_{\alpha,\beta}(x) = \frac{1}{\beta}$ so $inv_{\alpha,\beta}(x) = E_{\alpha,\beta}(-E^{-1}_{\alpha,\beta}(x))$.
- Closure with respect to the operation and the inverse:
  The range of $E_{\alpha,\beta}$ are strictly positive numbers as was proven above, so $\oplus$ produces strictly positive numbers. The range of the inverse function is $\mathbb{R}$, which is (a subset of) the domain of $E_{\alpha,\beta}$, so the operation is well defined. For the same reasons, the group is closed with respect to the inverse operation.

■

As above, some constraints are needed when $\alpha = 0$ and $\beta \neq 0$, where the function is not continuous. In that case $E_{0,\beta}(z) = \frac{1}{(1-z)\Gamma(\beta)}$. The inverse function is $E^{-1}_{0,\beta}(z) = \frac{(\Gamma(\beta)z-1)}{\Gamma(\beta)z}$. The inverse element is $inv_{0,\beta}(x) = \frac{1}{(1+E^{-1}_{0,\beta}(z))\Gamma(\beta)} = \frac{z}{(2\Gamma(\beta)z-1)}$.

When $\alpha = 0$ and $\beta \neq 0$ the operation becomes

$$x \oplus_{0,\beta} y = \frac{xy}{x + y - \Gamma(\beta)xy}.$$

So, when $\beta = 1$, and $\alpha = 0$ we get $x \oplus_{0,1} y = x \oplus_0 y = \frac{xy}{x \circledast_0 y}$. This gives $(x \oplus_0 y)(x \circledast_0 y) = xy$.

When $\alpha = 1$ the operation is $x \oplus_{1,\beta} y = \Gamma(\beta)xy$.

Following the same approach as we did with $\circledast$, we can define the operation generated by iterated composition of $\oplus$, which is:

$$z = x \oslash_{\alpha,\beta} y = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x)y). \tag{5.2}$$

Notice that $x \oslash_{1,1} y = e^{(ln(x)y)} = x^y$.

This operation is asymmetric in its arguments, so it is non-commutative but distributive with respect to $\oplus$ on its first argument: $(x \oplus_{\alpha,\beta} y) \oslash_{\alpha,\beta} z = (x \oslash_{\alpha,\beta} z) \oplus_{\alpha,\beta} (y \oslash_{\alpha,\beta} z)$. We can define a new operation, which is indeed symmetric and distributive on both arguments:

$$z = x \odot_{\alpha,\beta} y = x \oslash_{\alpha,\beta} E^{-1}_{\alpha,\beta}(y) = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x)E^{-1}_{\alpha,\beta}(y)). \tag{5.3}$$

Following the above reasoning, it is easy to prove that $\odot_{\alpha,\beta}$ is an Abelian group, so $(\oplus_{\alpha,\beta}, \odot_{\alpha,\beta},\ \mathbb{R}^+)$ is a field.

Both groups $(\oplus_{\alpha,\beta},\ \mathbb{R}^+)$ and $(\odot_{\alpha,\beta},\ \mathbb{R}^+)$ have an associated transform by Pontryagin duality. First, $(\oplus_{\alpha,\beta},\ \mathbb{R}^+)$ has a multiplicative character $\chi(x) = e^{\gamma 2\pi i E^{-1}_{\alpha,\beta}(x)}$ and an invariant measure of a compact subset $\mu(S) = \int_S E^{-1'}_{\alpha,\beta}(x)\, dx = \int_S \frac{1}{E'_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x))}\, dx$.

Second, $(\odot_{\alpha,\beta},\ \mathbb{R}^+)$ has a multiplicative character $\chi(x) = e^{ln(E^{-1}_{\alpha,\beta}(x))\gamma 2\pi i} = E^{-1}_{\alpha,\beta}(x)^{\gamma 2\pi i}$ and an invariant measure of a compact subset $\mu(S) = \int_S \frac{E^{-1'}_{\alpha,\beta}(x)}{E^{-1}_{\alpha,\beta}(x)}\, dx = \int_S \frac{1}{E'_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x))E^{-1}_{\alpha,\beta}(x)}\, dx$.

In both cases we have used the fact that when $\alpha \leq \beta$ and $0 \leq \beta \leq 1$, but α and β are not both zero, $E^{-1'}_{\alpha,\beta}(x) = \frac{1}{E'_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(x))}$ according to the inverse function theorem.

6. Other transforms and γ-hyperbolic functions

γ-hyperbolic functions can be used to define a generalization of the transform analyzed in [28]. Indeed, γ-hyperbolic functions are the kernel of the transforms, because by (1.2) they are the solution of the differential equation

$$L_k F^{\gamma^k}_{k,n}(z) = (D^k - \gamma^k)F^{\gamma^k}_{k,n}(z) = 0,\ k,n \in \mathbb{N},\ k \geq n,\ \gamma \in \mathbb{C}. \tag{6.1}$$

(6.1) can be extended using fractional calculus with (1.9):

$$L_{\beta+k} F^{\gamma^{\beta+k}}_{\beta+k,\beta}(z) = (D^{\beta+k}_{0+} - \gamma^{\beta+k})F^{\gamma^{\beta+k}}_{\beta+k,\beta}(z) = 0,\ \beta \in \mathbb{R},\ k \in \mathbb{N},\ \gamma \in \mathbb{C}. \tag{6.2}$$

A different approach is to use the γ-hyperbolic matrix defined in [6], which is a pseudo-circulant matrix, obtained by multiplying by γ each element above the diagonal of a circulant matrix formed by cycling the second subindex of the α-hyperbolic functions. In [6] it was proven that $det[F^{\gamma}_n(z)] = 1$, and $[F^{\gamma}_n(x)][F^{\gamma}_n(y)] = [F^{\gamma}_n(x+y)]$. Pseudo-circulant matrices constitute Lie algebras and can be used to define Lie groups which have γ-hyperbolic functions as special functions [12].

7. Relation with analysis on time scales

The analysis on measure chains (i.e. time scales) is a fairly new scientific topic. It was introduced in 1988 by Stefan Hilger and Bernd Aulbach [29, 30] and constitutes an active area of research [31]. It combines the traditional areas of continuous and discrete analysis. It is customary to follow a similar procedure to the one followed above to define a group induced by the exponential functions which are solutions of ordinary differential equations. This group is then used to define a Laplace transform. In some sense this approach is more general than the one used here, but it does not define a Fourier transform. Also, by using the family of Mittag-Leffler functions, we generalize it in other ways. The two approaches can be considered complementary and in the intersection, they have remarkable similarities. In this sense they can benefit from cross-pollination. In particular, the geometrical interpretations given in this paper provide insight to the time scales approach.

The Laplace transform for measure chains is defined [32] by

$$\mathcal{L}\{x\} = \int_0^\infty x(t) e_{\ominus z}(\sigma(t))\Delta t, \qquad \text{for } z \in \mathcal{D}\{x\}, \tag{7.1}$$

where $\ominus z$ is the inverse in the group induced by the exponential, $z \Rsh -1$ with our notation. $\mathcal{D}\{x\}$ are all $z \in \mathbb{C}$ for which the integral exists and $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$. The domain of the regulated function $x : \mathbb{T} \to \mathbb{C}$ is a closed subset $\mathbb{T}$ of the real line. $\sigma(t) = inf\{r \in \mathbb{T} | r > t\}$ and the limits in the space has to be taken with respect to the relative topology of $\mathbb{T}$. The derivative is $f^\Delta(t) = \lim_{s\to t, s\in\mathbb{T}} \frac{f(\sigma(t))-f(s)}{\sigma(t)-s}$.

As examples of it

(1) Standard calculus: $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\mu(t) = 0$, $\ominus z = -z$, $e_\alpha(t) = e^{\alpha t}$, $\int\limits_0^\infty f(t)\,\Delta t = \int\limits_0^\infty f(t)\,dt$.

(2) Nabla calculus: $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t - 1$ $\mu(t) = -1$, $\ominus z = -\frac{z}{1-z}$, $e_\alpha(t) = (1 - \alpha)^t$, $\int\limits_0^\infty f(t)\,\Delta t = \sum\limits_{t=0}^\infty f(t)$.

For Nabla calculus, the derivative is $(\nabla f)(t) = f(t) - f(t-1)$, the backward difference and the Laplace transform is

$$\mathcal{L}\{x\} = \sum_{t=1}^\infty (1-s)^{t-1} x(t). \tag{7.2}$$

Note that, for $t \in \mathbb{R}$,

$$E_{\alpha,\beta}(x)^{-t} = E_{\alpha,\beta}(E^{-1}_{\alpha,\beta}(E_{\alpha,\beta}(x)^{-t})) = E_{\alpha,\beta}(x \Rsh -t) = E_{\alpha,\beta}((x \Rsh -1) \Rsh t) = E_{\alpha,\beta}(inv(x) \Rsh t),$$

which in time scale notation is $e_{\ominus x}(t)$. The exponential function of Nabla calculus $e_w(t) = E_0(w)^{-t} = E_0(w \Rsh -t)$ satisfies $e_{\ominus w}(t) = E_0(w \Rsh t)$. The Fourier transform associated with $E_0(x)$ is

$$\hat{f}(\beta) = \int_{-\infty}^{1} f(x)(1-x)^{\beta 2\pi i - 1}\, dx - \int_{1}^{\infty} f(x)(1-x)^{\beta 2\pi i - 1}\, dx.$$

If f is defined in $[1, +\infty)$, then

$$\hat{f}(\beta) = -\int_{1}^{\infty} f(x)(1-x)^{\beta 2\pi i - 1}\, dx. \tag{7.3}$$

With the change of variables $s = 2\pi\beta i$ (7.3) can be interpreted as a continuous form of (7.2) albeit negated (which does not change any of its properties). It is reasonable that the exponential associated with hypercubes, which are the balls of the taxi-cab distance, appears naturally in a discrete setting.

This approach allows to go from our Fourier transform to a Laplace transform by setting $s = 2\pi\beta i$ and the limits of integration from the identity element of the group to infinity,

$$\mathcal{L}\{f\}(s) = \int_{id}^{\infty} f(x) E_{\alpha,\beta}(x)^{-s-1} E'_{\alpha,\beta}(x)\, dx = \int_{id}^{\infty} f(x) E_{\alpha,\beta}(inv(x) \Rsh (s+1)) E'_{\alpha,\beta}(x)\, dx.$$

This relation works in both directions, and Pontryagin duality can be used to define rigorously a Fourier transform in a measure chain too. It is based on the group defined by the exponential, which is related to the Laplace transform in that space. Mackey [33] defined a theory for the Laplace transform on LCA groups, which matches the above definitions of generalized characters and allows us to interpret these results in the same framework, generalizing Pontryagin duality.

Based on our geometrical interpretations, we are free to define other exponential functions with a good behaviour on discrete settings. On one hand we have Nabla calculus, which corresponds to E_0 and, on the other hand, classical discrete calculus, which corresponds to E_1 with regular exponentials and the classical discrete Fourier (DFT) and Laplace transforms. We can define an exponential induced by the discrete volume defined by the intersection of the measure polytope and the cross-polytope (rectification). For example, in three dimensions, this would produce a cuboctahedron. The function defined by this infinite sum of volumes replaces the $erfcx(-x)$ function, associated to the sphere in the continuous setting.

The exponential for Nabla calculus is $E_0(x) = \frac{1}{1-x}$, obtained from the backward difference and the one for delta calculus is $E_0(-x) = \frac{1}{1+x}$ obtained from the forward difference. By analogy, can use $E_{\alpha,\beta}(-x)$ to give another version of all the groups defined above. For example, we can define $a \circledast^{\alpha,\beta} b = -E^{-1}_{\alpha,\beta}(E_{\alpha,\beta}(-x)E_{\alpha,\beta}(-y))$, which is also a group with similar properties to $\circledast_{\alpha,\beta}$. For example $a \circledast^{0,1} b = a + b + ab$.

We can also generalize the Mittag-Leffler polynomials. Mittag-Leffler polynomials [34, 35] $g_n(y)$ are the coefficients of the expansion

$$\left(\frac{1+x}{1-x}\right)^y = \sum_{n=0}^{\infty} \frac{g_n(y)x^n}{n!} = \left(\frac{E_0(x)}{E_0(-x)}\right)^y \qquad |x| < 1. \tag{7.4}$$

Thus, we define the generalized Mittag-Leffler polynomials,

$$\left(\frac{E_{\alpha,\beta}(x)}{E_{\alpha,\beta}(-x)}\right)^y = \sum_{n=0}^{\infty} \frac{g_{n,\alpha,\beta}(y)x^n}{n!}, \qquad x \in \mathbb{R}, \alpha, \beta \in \mathbb{R}, \alpha \neq 0. \tag{7.5}$$

Assuming both α and β are not simultaneously zero, we can obtain explicit expressions of such polynomials by calculating the Taylor series of the definition around zero, for example

$$g_{0,\alpha,\beta}(y) = 1, g_{1,\alpha,\beta}(y) = \frac{2y}{\Gamma(\beta+\alpha)}, \quad g_{2,\alpha,\beta}(y) = \frac{4y^2}{\Gamma(\beta+\alpha)^2},$$

$$g_{3,\alpha,\beta}(y) = \frac{8y^3}{\Gamma(\beta+\alpha)^3} + 4y\left(\frac{1}{\Gamma(\beta+\alpha)^3} + \frac{3}{\Gamma(\beta+3\alpha)} - \frac{3}{\Gamma(\beta+\alpha)\Gamma(\beta+2\alpha)}\right),$$

$$g_{4,\alpha,\beta}(y) = \frac{16y^4}{\Gamma(\beta+\alpha)^4} + 32y^2\left(\frac{1}{\Gamma(\beta+\alpha)^4} + \frac{3}{\Gamma(\beta+3\alpha)\Gamma(\beta+\alpha)} - \frac{3}{\Gamma(\beta+\alpha)^2\Gamma(\beta+2\alpha)}\right).$$

For the special case of the exponentials, $g_{n,1,1}(y) = (2y)^n$.

We don't know yet if the orthogonality relations of regular Mittag-Leffler polynomials extend in some way to this generalized polynomials into the family of generalized hypergeometric polynomials. These generalized Mittag-Leffler polynomials can be deformed, like the classical ones as in [34].

Another point of contact between the two approaches comes from the deformed exponentials as defined in [37]. The deformed exponentials are $e_h(x,y) = e^{\frac{y}{h}ln(1+hx)}$ which coincides with $(1+xh) \oslash_1 \frac{y}{h}$. The operation $\oslash$ is the exponential operation defined by the group $\oplus$ [1], which explains many of the properties analyzed in [37].

8. Recursive group definition and beyond

We have now several groups, $\circledast$, $\oplus$, $\odot$ and their various counterparts for reflected Mittag-Leffler functions. These groups are defined by formulas like $a \circ b = E^{-1}(E(a) \star E(b))$ with $a, b \in \mathbb{R}$ and $a \diamond b = E(E^{-1}(a) \bullet E^{-1}(b))$ with $a, b \in \mathbb{R}^+$. For example, if $\circ = \circledast_\alpha$, then $E = E_\alpha$ and $\star$ is the regular multiplication. We could apply these formulas recursively to define new operations. We are going to denote these operations with the concatenation of the symbols of the operations used. If we combine $\circledast_\alpha$ and $\oplus_\beta$, then

$$a \circledast_\alpha \oplus_\beta b = E_\alpha^{-1}(E_\alpha(a) \oplus_\beta E_\alpha(b)) = E_\alpha^{-1}(E_\beta(E_\beta^{-1}(E_\alpha(a)) + E_\beta^{-1}(E_\alpha(b)))).$$

As long as we alternate $\circ$ and $\diamond$ operations, we can define an new operation out of a chain of operations. Combining the different operations, we obtain

$$\circledast_{\alpha,\beta} \oplus_{\alpha,\beta} = \circledast_{1,1} = +, \quad \oplus_{\alpha,\beta} \circledast_{\alpha,\beta} = \oplus_{1,1} = *,$$

$$\circledast_{\alpha,\beta} * = \circledast_{\alpha,\beta}, \quad \oplus_{\alpha,\beta} + = \oplus_{\alpha,\beta}, \quad \oplus_{\alpha,\beta} * = \odot_{\alpha,\beta}.$$

[1]Strictly it is the operation defined by iterated composition.

Taking into account $\circledast_{1,1} = +$ and $\oplus_{1,1} = *$, the regular addition and multiplication act as the even and odd right identities, respectively, and $\circledast_{\alpha,\beta}$ and $\oplus_{\alpha,\beta}$ with the same indices are the inverse of each other.

Note that not all combinations are possible. For example, $a \circledast_{\alpha,\beta} + b = E_\alpha^{-1}(E_\alpha(a) + E_\alpha(b))$ is not a group, due to the lack of identity, because there is no $b \in \mathbb{R}$ such that $E_\alpha(b) = 0$.

We can also combine them with reflected functions, for which we obtain

$$\begin{aligned}
&\circledast^{\alpha,\beta} \oplus^{\alpha,\beta} = \circledast^{1,1} = +, \quad \oplus^{\alpha,\beta} \circledast^{\alpha,\beta} = \oplus^{1,1} = *,\\
&\circledast^{\alpha,\beta} \oplus_{\alpha,\beta} = \circledast^{1,1} = +, \quad \oplus^{\alpha,\beta} \circledast_{\alpha,\beta} = \oplus^{1,1} = *,\\
&\circledast^{\alpha,\beta} * = \circledast^{\alpha,\beta}, \quad \oplus^{\alpha,\beta} + = \oplus^{\alpha,\beta}, \quad \oplus^{\alpha,\beta} * = \odot^{\alpha,\beta}.
\end{aligned}$$

The special cases $a \circledast_{1,1} \oplus_{0,1} b = ln\left(\frac{e^{a+b}}{e^a+e^b-e^{ab}}\right)$ and $a \oplus^{0,1} b = a \oplus_{0,1} b = \frac{ab}{a+b-ab} = \frac{ab}{a \circledast_0 b}$, appear.

Note that these relations need to be applied from right to left, since the domain is defined by the left-most element in the chain, thus $\circledast^{\alpha,\beta}(\underbrace{\oplus^{\alpha,\beta} \circledast^{\alpha,\beta}}_{\text{correct}}) = \circledast^{\alpha,\beta} * = \circledast^{\alpha,\beta}$ is correct, but $(\underbrace{\circledast^{\alpha,\beta} \oplus^{\alpha,\beta}}_{\text{incorrect}}) \circledast^{\alpha,\beta} = + \circledast^{\alpha,\beta} \neq \circledast^{\alpha,\beta}$ is not right.

Note also that, in the same way that $\oplus^{\alpha,\beta} * = \odot^{\alpha,\beta}$ and $\oplus^{\alpha,\beta}$ form a field in $\mathbb{R}^+$, many other similar combinations are possible. For example, the operations $(\oplus^{\alpha,\beta} \circledast^{\gamma,\delta} \oplus^{\alpha,\beta} *, \oplus^{\alpha,\beta} \circledast^{\gamma,\delta} \oplus^{\alpha,\beta}, \mathbb{R}^+)$ also constitute a field.

9. Conclusions and future work

(1.5), together with the fact that the Mittag-Leffler function is the sum of volumes of generalized n-balls, gives us a geometrical interpretation of the fractional derivative.

In this context, the integer derivative is the operator which has as an eigenfunction the sum of the volumes of cross-polytopes of constant radius as eigenvalue, so the fractional derivative is the operator which has as an eigenfunction the sum of the volumes of generalized n-balls with radius $R^{\frac{1}{\alpha}}$ except for the constant term. The far-reaching consequences of this geometrical interpretation is a work in progress. For example, instead of measure polytopes and cross-polytopes, we could consider the volumes of simplices and other n-dimensional families of bodies.

As a future work, it would also be interesting to explore the extension of matrix groups induced by the above groups.

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