On the Spectrum of two-layer approach and Multiplex PageRank

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Abstract

In this paper, we present some results about the spectrum of the matrix associated with the computation of the *Multiplex PageRank* defined by the authors in a previous paper. These results can be considered as a natural extension of the known results about the spectrum of the *Google* matrix. In particular, we show that the eigenvalues of the transition matrix associated with the multiplex network can be deduced from the eigenvalues of a block matrix containing the stochastic matrices defined for each layer. We also show that, as occurs in the classic PageRank, the spectrum is not affected by the personalization vectors defined on each layer but depends on the parameter α that controls the teleportation. We also give some analytical relations between the eigenvalues and we include some small examples illustrating the main results.

Keywords: PageRank, centrality measures, Multiplex networks

1. Introduction

This paper deals with two main topics: Multiplex networks and PageRank. A Multiplex network can be considered as a network formed by several layers (networks) with the same nodes, but different topology inside of each layer. A typical example of a Multiplex network consists of the set of several networks formed by the same people connected on different Social Networks like Facebook, Twitter, Linkedin, etc. Among the applications of Multiplex Networks we can

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cite the following: In [24], Multiplex networks are applied to study the coupling between the street network and the subway system for the metropolitan areas of London and New York; in [20], the authors consider the different lines of the Madrid Metro system to form a Multiplex network; in [5], the authors apply Multiplex networks to consider the different layers of Wikipedia that cite some scientists and philosophers. In the same paper, the authors form a Multiplex network by considering some airports as the nodes and some airline companies to define the layers. Some other applications can be found in [2].

The other topic of the paper is PageRank, which is a standard centrality measure that can be defined on a network. Several new research works about the PageRank algorithm, originally devised by the founders of Google [18], appear in the literature with the aim of improving the numerical performance of the method as well as the range of applications. For example, in the latest research papers one can find new numerical methods for computing PageRank (see, e.g., [25], [27], [23], [17], [10]) and new applications (see, e.g., [1], [22], [12], [15], and the dedicated paper [8]) including some applications related to the emerging topic of multiplex networks, like the studies in [2] and [5]. New generalizations of PageRank to usual networks also include the use of higher-order Markov chains. We recall that PageRank can be considered as a stationary state of a (1-order) Markov chain that transforms the state of a system with the knowledge of the previous state. In a network of n nodes, this is modeled by considering a transition stochastic matrix of size $n \times n$. On the contrary, when considering a higher order Markov chain, the changes in the state of the nodes can be modeled by using a *stochastic matrix* of size $n \times n^r$, being r the number of previous states to be considered. In [9] it is shown how to apply this formalism to define a new PageRank called *Multilinear PageRank* (that is, in fact a monoplex PageRank since it is applied on only one network). In this paper we only use 1-order Markov chains.

Our interest in the PageRank algorithm was motivated by the property of the algorithm to bias the PageRank -and therefore the resulting ranking- to some preferred nodes. This biasing is done by means of the so-called personalization vector, see [14], [3]. In [7] it is shown that the biasing produced by the personalization vector $\mathbf{v}$ is a limited one; the PageRank score of each node can only attain values inside a precise subinterval of $(0, 1)$ depending on the entries of a certain matrix. Given that the increasing interest in *Multiplex Networks* (see, e.g., [4], [13]) originated some generalizations of the concept of (classic, *monoplex*) PageRank to *Multiplex* PageRank, it was natural to extend the study on the effect of the personalization vectors in this new framework.

There are different ways to define Multiplex PageRank (see, e.g., [11], [5], [6]). In this paper we use the definition introduced in [20]. According to this approach, the Multiplex PageRank is the unitary positive eigenvector of a certain stochastic matrix M_k associated with the multiplex network equipped with k layers with n nodes on each layer (it is worth noting that M_k takes into account the personalization vectors for each layer). In the same manner as in the monoplex PageRank, the Multiplex PageRank may change when there exists a change in the personalization vectors. In [19] we showed the type of dependence of the Multiplex PageRank on the personalization vectors, as well as the precise bounds for the multiplex PageRank, in a similar fashion as was done in [7] for the monoplex PageRank.

In this paper we focus on the spectral properties of matrix M_k. This study can be considered as a natural extension of the existing studies about classic PageRank focused on the spectrum of the so-called *Google* matrix (see, e.g., [14], [26], [3]). It turns out that the definition of matrix M_k is related with a particular interpretation of the classic PageRank in terms of two layers (we called this concept the *two-layer approach PageRank*, see [20]); this procedure is formulated by introducing a certain transition matrix M_A (where A denotes the adjacency matrix of the monoplex network). As a consequence, when considering a multiplex network composed of k layers, the matrix M_k is related to a block matrix, denoted by $\mathbb{B}_{1,1}$, that takes into account the adjacency matrices A_i associated with each layer identified by $i = 1, \dots, k$. This paper is, therefore, focused on the properties of the eigenvalues of M_k and $\mathbb{B}_{1,1}$. The study of the eigenspaces of these matrices is beyond our objectives but it could be done by following the techniques shown in [21].

The structure of the paper is as follows. In section 2 we establish the notation and recall some known results. In section 3 we show the spectrum of the matrix M_A as well as some properties derived from the main theorem in this section. In section 4 we give the spectrum of the matrix M_k associated with the multiplex PageRank, in terms of the spectrum of the block matrix $\mathbb{B}_{1,1}$.

2. Notation and some preliminary definitions

We recall some notation from [7] and [20]. Vectors of $\mathbb{R}^{n\times 1}$ will be denoted by column matrices and we will use the superscript T to indicate matrix transposition. The vector of $\mathbb{R}^{n\times 1}$ with all its components equal to 1 will be denoted by $\mathbf{e}$. That is, $\mathbf{e} = (1, \cdots, 1)^T$.

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ be a directed graph where $\mathcal{N} = \{1, 2, \dots, n\}$ and $n \in \mathbb{N}$. The pair (i, j) belongs to the set $\mathcal{E}$ if and only if there exists a link connecting node i to node j. The *adjacency matrix* of $\mathcal{G}$ is an $n \times n$-matrix

$$A = (a_{ij}) \text{ where } a_{ij} = \begin{cases} 1, & \text{if } (i, j) \text{ is a link of } \mathcal{G} \\ 0, & \text{otherwise.} \end{cases}$$

A link (i, j) is said to be an *outlink* for node i and an *inlink* for node j. We denote $k_{out}(i)$ the *outdegree* of node i, i.e., the number of outlinks of a node i. Notice that $k_{out}(i) = \sum_k a_{ik}$. The graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ may have *dangling nodes*, which are nodes $i \in \mathcal{N}$ with zero outdegree. Dangling nodes are characterized by a vector $\mathbf{d} \in \mathbb{R}^{n\times 1}$ with components d_i defined by

$$d_i = \begin{cases} 1, & \text{if } i \text{ is a dangling node of } \mathcal{G} \\ 0, & \text{otherwise.} \end{cases}$$

Let $P_A = (p_{ij}) \in \mathbb{R}^{n\times n}$ be the *row stochastic matrix* associated with $\mathcal{G}$ defined in the following way:

- if i is a dangling node, $p_{ij} = 0$ for all $j = 1, \dots, n$,
- otherwise, $p_{ij} = \frac{a_{ij}}{k_{out}(i)} = \frac{a_{ij}}{\sum_k a_{ik}}$.

Note that each coefficient p_{ij} can be considered as the probability of jumping from the node i to the node j.

We recall that one of the features of the personalized PageRank algorithm is that some *extra* probability of jumping is given to any node. This extra or *teleportation* probability is assigned by using a *personalization vector* $\mathbf{v}$, which is a probability distribution vector. If, in addition, the graph has dangling nodes then the algorithm needs to assign an additional probability of jumping to these dangling nodes; this is done by introducing a probability distribution vector $\mathbf{u}$. With these ingredients, plus a teleportation parameter α, we have everything to build a primitive and stochastic matrix, called *Google matrix*, that we denote by G.

Formally, $G = G(\alpha, \mathbf{u}, \mathbf{v})$, with $\alpha \in (0, 1)$, is defined as

$$G = \alpha(P_A + \mathbf{d}\mathbf{u}^T) + (1 - \alpha)\mathbf{e}\mathbf{v}^T \in \mathbb{R}^{n \times n}. \tag{2.1}$$

Note that G is row-stochastic, i.e., $G\mathbf{e} = \mathbf{e}$. Recall that $\mathbf{v} \in \mathbb{R}^{n \times 1}$, with $\mathbf{v} > 0$ and $\mathbf{v}^T\mathbf{e} = 1$. Analogously, $\mathbf{u} \in \mathbb{R}^{n \times 1}$ such that $\mathbf{u} > 0$ and $\mathbf{u}^T\mathbf{e} = 1$.

The *spectrum* of a square matrix M is the set of all its eigenvalues that will be denoted by $\sigma(M)$ (see, for example [16]).

The *PageRank vector* $\pi = \pi(\alpha, \mathbf{u}, \mathbf{v})$ is the unique positive eigenvector of G^T associated to eigenvalue 1 such that $\pi^T\mathbf{e} = 1$, i.e., $\pi > 0$, $\pi^T\mathbf{e} = 1$ and $\pi^T G = \pi^T$ (see [18]). Since we focus our interest in $\mathbf{v}$ we also refer to π as the *personalized PageRank vector*.

Note also that from (2.1) we easily have

$$\pi^T = \alpha\pi^T(P_A + \mathbf{d}\mathbf{u}^T) + (1 - \alpha)\mathbf{v}^T. \tag{2.2}$$

We will assume throughout the paper that $\mathbf{d} = \mathbf{0}$ because similar results to those presented here can be straightforwardly obtained when $\mathbf{d} \neq \mathbf{0}$.

We recall the definition of two-layer approach PageRank $\hat{\pi}_A$ introduced in [20].

Definition 2.3. Given an adjacency matrix A we define the two-layer approach PageRank of A and denote it by $\hat{\pi}_A$ as the vector $\hat{\pi}_A = \pi_u + \pi_d \in \mathbb{R}^{n \times 1}$, where $\hat{\pi}_M^T = [\pi_u^T \quad \pi_d^T] \in \mathbb{R}^{1 \times 2n}$ is the unique vector that satisfies:

(i) $\hat{\pi}_M^T = \hat{\pi}_M^T M_A$ with $\hat{\pi}_M^T\mathbf{e} = 1$,

(ii) $\pi_u^T\mathbf{e} = \alpha$, $\pi_d^T\mathbf{e} = 1 - \alpha$,

where M_A is the row stochastic matrix associated to A given by

$$M_A = \begin{pmatrix} \alpha P_A & (1 - \alpha)I_n \\ \alpha I_n & (1 - \alpha)\mathbf{e}\mathbf{v}^T \end{pmatrix} \in \mathbb{R}^{2n \times 2n}. \tag{2.4}$$

An extension of this concept to the case of multiplex networks was presented in [20], as follows. Given a Multiplex network $\mathcal{M} = (\mathcal{N}, \mathcal{E}, \mathcal{S})$, with layers $\mathcal{S} = \{\ell_1, \ldots, \ell_k\}$ whose adjacency matrices are $A_1, \ldots, A_k \in \mathbb{R}^{n\times n}$ and some personalized vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^{n\times 1}$, then we take

$$M_k = \frac{1}{k}\begin{pmatrix} \mathbb{B}_{1,1} & \mathbb{B}_{1,2} \\ \mathbb{B}_{2,1} & \mathbb{B}_{2,2} \end{pmatrix} \in \mathbb{R}^{2kn\times 2kn}, \tag{2.5}$$

where

$$\mathbb{B}_{1,1} = \begin{pmatrix} \alpha P_{A_1} & I_n & \cdots & I_n \\ I_n & \alpha P_{A_2} & \cdots & I_n \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \cdots & \alpha P_{A_k} \end{pmatrix}, \tag{2.6}$$

$$\mathbb{B}_{2,2} = (1-\alpha)\begin{pmatrix} \mathbf{e}\mathbf{v}_1^T & \cdots & \mathbf{e}\mathbf{v}_k^T \\ \vdots & \ddots & \vdots \\ \mathbf{e}\mathbf{v}_1^T & \cdots & \mathbf{e}\mathbf{v}_k^T \end{pmatrix}, \tag{2.7}$$

$$\mathbb{B}_{1,2} = (1-\alpha)I_{kn} \in \mathbb{R}^{kn\times kn}, \qquad \mathbb{B}_{2,1} = k\alpha I_{kn} \in \mathbb{R}^{kn\times kn}. \tag{2.8}$$

Definition 2.9. The Multiplex PageRank $\hat{\pi}_k$ is the vector

$$\hat{\pi}_k = \frac{1}{k}\left(\pi_{u1} + \pi_{u2} + \cdots + \pi_{uk} + \pi_{d1} + \pi_{d2} + \cdots + \pi_{dk}\right) \in \mathbb{R}^{n\times 1},$$

where $\hat{\pi}_M^T = [\pi_{u1}^T \quad \pi_{u2}^T \quad \ldots \quad \pi_{uk}^T \quad \pi_{d1}^T \quad \pi_{d2}^T \quad \ldots \quad \pi_{dk}^T]$ is the unique vector that satisfies:

(i) $\hat{\pi}_M^T = \hat{\pi}_M^T M_k$ with $\hat{\pi}_M^T\mathbf{e} = k$,

(ii) $\pi_{ui}^T\mathbf{e} = \gamma$, $\pi_{di}^T\mathbf{e} = 1-\gamma$ for all $i = 1, 2, \ldots, k$ and $\gamma = \frac{k\alpha}{1+\alpha(k-1)}$.

3. Spectrum of the matrix of the two-layer approach PageRank

Spectral properties of matrix G (see (2.1)) are well established (see, for example [3]). In particular, it is known that the spectrum of G and the spectrum of P_A are deeply related [14], as the following result shows:

Theorem 3.1 ([14], Thm.5.1)**.** *Following the notation of the previous section, if* $\sigma(P_A) = \{1, \mu_2, \mu_3, \ldots, \mu_k\}$*, then* $\sigma(G) = \{1, \alpha\mu_2, \alpha\mu_3, \ldots, \alpha\mu_k\}$*.*

If we consider the two-layer approach of PageRank or the Multiplex PageRank defined before, it is natural to look for similar results relating the spectrum of the involved stochastic matrices. In this section we will focus on the two-layer approach case and next section is devoted to the Multiplex PageRank setting. Therefore, the main goal of this section is relating $\sigma(P_A)$ with $\sigma(M_A)$.

First, note that if $\lambda \in \sigma(M_A)$, there exists a nonzero eigenvector $[\mathbf{u}^T\, \mathbf{w}^T]$ associated to λ, i.e., satisfying

$$[\mathbf{u}^T\, \mathbf{w}^T]M_A = \lambda[\mathbf{u}^T\, \mathbf{w}^T].$$

Hence

$$\left.\begin{array}{rcl} \mathbf{u}^T\alpha P_A + \alpha\mathbf{w}^T & = & \lambda\mathbf{u}^T \\ (1-\alpha)\mathbf{u}^T + (1-\alpha)\mathbf{w}^T\mathbf{e}\mathbf{v}^T & = & \lambda\mathbf{w}^T \end{array}\right\} \tag{3.2}$$

Note that since P_A is row stochastic, then $1 \in \sigma(P_A)$. By Definition 2.3, it is clear that $\lambda = 1$ is the dominant eigenvalue of M_A (note furthermore that M_A is row-stochastic, irreducible and primitive and therefore the eigenvalue 1 is simple and the unique with norm equal to the unity).

Lemma 3.3. *The eigenvalues of P_A can be classified into two sets:*

(i) Those that have eigenvectors $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{e} \neq 0$. The only element of this set is $\mu = 1$.

(ii) Those that have eigenvectors $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{e} = 0$. This happens for the rest of the eigenvectors of P_A, including those associated to $\mu = 1$ if it is multiple. If $\mu = 1$ has multiplicity q, in this set we have $\mu = 1$ (with multiplicity $q-1$) and the rest of eigenvalues μ_i.

Proof. Let μ be an eigenvalue of P_A. If $\mu \neq 1$, let $\mathbf{u}$ be an associated left eigenvector $\mathbf{u}$, i.e., a nonzero vector such that $\mathbf{u}^T P_A = \mu\mathbf{u}^T$. Multiplying by $\mathbf{e}$ on the right and using that $\mu \neq 1$ we obtain that $\mathbf{u}^T\mathbf{e} = 0$.

Now, suppose that $\mu = 1$. Notice that $\mu = 1$ coincides with the spectral radius of P_A, so there exists an associated left eigenvector $\widehat{\mathbf{u}}$ such that $\widehat{\mathbf{u}}^T\mathbf{e} \neq 0$. We consider the following two possibilities:

(1) If $\mu = 1$ is a simple eigenvalue of P_A, its associated left eigenvectors must satisfy $\mathbf{u}^T\mathbf{e} \neq 0$, since they all belong to the one-dimensional eigenspace generated by $\widehat{\mathbf{u}}^T$.

(2) If $\mu = 1$ is a multiple eigenvector with algebraic multiplicity q, then the eigenspace associated to 1 has dimension q because P_A is row stochastic, see [16, p. 696]. Let $\mathbf{u}_i$, $i = 1, \ldots, q$, be q linearly independent eigenvectors associated to 1. Since $\widehat{\mathbf{u}}^T$ belongs to this eigenspace and $\widehat{\mathbf{u}}^T\mathbf{e} \neq 0$, we can suppose without loss of generality that $\mathbf{u}_q^T\mathbf{e} \neq 0$. Replace each $\mathbf{u}_i$, $i = 1, \ldots, q-1$, by

$$\tilde{\mathbf{u}}_i = \mathbf{u_i} - \frac{\mathbf{u_i}^T\mathbf{e}}{\mathbf{u}_q^T\mathbf{e}}\mathbf{u}_q, \qquad i = 1, \ldots, q-1.$$

Then $\tilde{\mathbf{u}}_1, \ldots, \tilde{\mathbf{u}}_{q-1}$ are $q-1$ linearly independent eigenvectors associated to 1 and all of them satisfy $\tilde{\mathbf{u}}_i^T\mathbf{e} = 0$.

Therefore, if an eigenvector $\mathbf{u}$ associated to μ satisfies $\mathbf{u}^T\mathbf{e} \neq 0$, then $\mu = 1$. If $\mathbf{u}^T\mathbf{e} = 0$, then either $\mu \neq 1$ or $\mu = 1$ is a multiple eigenvalue. □

Now we will focus on the spectrum of M_A with a series of technical lemmas.

Lemma 3.4. *Let $0, 1 \neq \lambda \in \sigma(M_A)$ and $[\mathbf{u}^T \ \mathbf{w}^T]$ be a (left) eigenvector associated to λ. Then, $\mathbf{u}^T\mathbf{e} = 0$ and $\mathbf{w}^T\mathbf{e} = 0$.*

Proof. By multiplying equation (3.2) by $\mathbf{e}$ on the right, we get that

$$\left.\begin{array}{rcl}\alpha(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e} & = & \lambda \mathbf{u}^T\mathbf{e} \\ (1-\alpha)(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e} & = & \lambda \mathbf{w}^T\mathbf{e}\end{array}\right\} \tag{3.5}$$

Now, adding up last two equations we obtain that

$$(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e} = \lambda(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e}. \tag{3.6}$$

Note that $(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e} \in \mathbb{R}$ and since $\lambda \neq 1$ the unique solution of (3.6) is

$$(\mathbf{u}^T + \mathbf{w}^T)\mathbf{e} = 0.$$

Now, if we plug the last expression into the first equation in (3.5) we get that $\mathbf{u}^T\mathbf{e} = 0$, since $\lambda \neq 0$ and thus also $\mathbf{v}^T\mathbf{e} = 0$. $\square$

Lemma 3.7. $\lambda = 0$ *is an eigenvalue of* M_A.

Proof. Let

$$\mathbf{z} = \begin{pmatrix} \mathbf{e} \\ \frac{-\alpha}{1-\alpha}\mathbf{e} \end{pmatrix} \in \mathbb{R}^{2n\times 1}.$$

Since P_A is row stochastic and $\mathbf{e}^T\mathbf{v} = 1$, then we have

$$\mathbf{z}^T M_A^T = [\mathbf{e}^T \;\; \frac{-\alpha}{1-\alpha}\mathbf{e}^T] \begin{pmatrix} \alpha P_A^T & \alpha I_n \\ (1-\alpha)I_n & (1-\alpha)\mathbf{v}\mathbf{e}^T \end{pmatrix} = [\mathbf{0}_{1\times n} \; \mathbf{0}_{1\times n}] = 0\,\mathbf{z}^T$$

and thus $M_A\,\mathbf{z} = 0\,\mathbf{z}$. $\square$

By using the previous Lemmas, we get the main result of this section relating $\sigma(P_A)$ with $\sigma(M_A)$.

Theorem 3.8. *If* $\sigma(P_A) = \{1, \mu_2, \mu_3, \ldots, \mu_k\}$, *then*

$$\sigma(M_A) = \{1, 0\} \cup \{\lambda_i^+, \lambda_i^-;\;\; 1 \le i \le k\},$$

where

$$\lambda_i^+ = \frac{\alpha}{2}\left(\mu_i + \sqrt{\mu_i^2 - 4 + \frac{4}{\alpha}}\right), \quad \lambda_i^- = \frac{\alpha}{2}\left(\mu_i - \sqrt{\mu_i^2 - 4 + \frac{4}{\alpha}}\right). \tag{3.9}$$

Proof. Since P_A and M_A are row stochastic it is straightforward that $1 \in \sigma(P_A)$ and $1 \in \sigma(M_A)$. From Lemma 3.7, $0 \in \sigma(M_A)$. Now, let us show that any $0, 1 \neq \lambda \in \sigma(M_A)$ is of the form

$$\lambda = \frac{\alpha\mu \pm \sqrt{\alpha^2\mu^2 + 4\alpha(1-\alpha)}}{2} = \frac{\alpha}{2}\left(\mu \pm \sqrt{\mu^2 - 4 + \frac{4}{\alpha}}\right)$$

for some $\mu \in \sigma(P_A)$. Let $[\mathbf{u}^T \; \mathbf{w}^T]$ be a (left) eigenvector associated to such λ. From Lemma 3.4 we know that $\mathbf{w}^T\mathbf{e} = 0$, and therefore from the second equation of (3.2) we get

$$\mathbf{w}^T = \frac{1-\alpha}{\lambda}\mathbf{u}^T.$$

By plugging last expression into the first equation of (3.2) we get

$$\mathbf{u}^T \alpha P_A + \alpha \frac{1-\alpha}{\lambda} \mathbf{u}^T = \lambda \mathbf{u}^T,$$

that can be rewritten as

$$\mathbf{u}^T P_A = \left(\frac{\lambda}{\alpha} - \frac{1-\alpha}{\lambda} \right) \mathbf{u}^T. \tag{3.10}$$

Therefore, we conclude that

$$\mu := \left(\frac{\lambda}{\alpha} - \frac{1-\alpha}{\lambda} \right) \tag{3.11}$$

is an eigenvalue of P_A associated to the eigenvector $\mathbf{u}^T$.

By manipulating (3.11) we get to

$$\lambda^2 - \lambda\alpha\mu - \alpha(1-\alpha) = 0$$

and therefore

$$\lambda = \frac{\alpha\mu \pm \sqrt{\alpha^2\mu^2 + 4\alpha(1-\alpha)}}{2} = \frac{\alpha}{2} \left(\mu \pm \sqrt{\mu^2 - 4 + \frac{4}{\alpha}} \right).$$

For the converse let any μ be in the spectrum of P_A with associated (left) eigenvector $\mathbf{u}^T$ such that $\mathbf{u}^T\mathbf{e} = 0$. Define the second degree polynomial on the variable x

$$m(x) = x^2 - \alpha\mu x - \alpha(1-\alpha)$$

and let λ be a root of $m(x)$. Since $\lambda(1-\lambda) \neq 0$, it is clear that $\lambda \neq 0$ and hence we can define the vector

$$\mathbf{w}^T := \frac{1-\alpha}{\lambda} \mathbf{u}^T.$$

It is easy to see that $[\mathbf{u}^T\, \mathbf{w}^T]$ satisfies equation (3.2) and therefore it is an eigenvector of M_A associated to λ. □

Remark 3.12. Note that, as in the classic PageRank, the eigenvalues of M_A do not depend on the choice of the personalization vector $\mathbf{v}$.

Remark 3.13. Note that Theorem 3.8 provides a method to compute the spectrum of M_A from the spectrum of P_A: apart from the eigenvalue 1, which always belongs to $\sigma(M_A)$, any eigenvalue $\mu_i \neq 1$ of P_A gives rise to two eigenvalues λ_i^+ and λ_i^- of M_A. Moreover, if 1 is a multiple eigenvalue then also

$$\frac{\alpha}{2} \left(1 \pm \sqrt{\frac{4}{\alpha} - 3} \right) \in \sigma(M_A).$$

Hence Theorem 3.8 allows computing the spectrum of a $2n \times 2n$ matrix (M_A) in terms of the spectrum of a smaller $n \times n$ matrix (P_A), so the computational complexity decreases its order.

Remark 3.14. Note that $\lambda_i^+ + \lambda_i^- = \alpha\mu_i$ for every $2 \le i \le k$ and therefore these sums together with $\lambda = 1$ is in fact the spectrum for matrix G in the classic PageRank given by Theorem 3.1. This property shows that the two-layer PageRank is an extension of classic PageRank that spreads -in a beautiful simple way- the spectrum.

Given that the pair of complex eigenvalues λ_i^+ and λ_i^- in Theorem 3.8 are the solutions of a quadratic equation (3.9), we have the following straightforward result.

Corollary 3.15. *If λ_i^+ and λ_i^- are two eigenvalues of M_A given by equation* (3.9) *in Theorem 3.8 then it holds that*

$$\lambda_i^+ \lambda_i^- = \alpha(\alpha - 1) < 0$$

for all $i = 2, \ldots, k$.

Finally, we can state one more property about correspondence between the eigenvalues of P_A and M_A.

Corollary 3.16. *If μ_i and μ_j are complex conjugate eigenvalues of P_A then λ_i^+ and λ_j^+ are complex conjugate eigenvalues of M_A. The same holds for λ_i^- and λ_j^-.*

Proof. By hypothesis we have that $\mu_i = \overline{\mu_j}$. Then it is clear that $\mu_i^2 = \overline{\mu_j^2}$ and therefore

$$\sqrt{\mu_i^2} = \sqrt{\overline{\mu_j^2}} = \overline{\sqrt{\mu_j^2}}.$$

Thus we also have that

$$\sqrt{\mu_i^2 - 4 + \frac{4}{\alpha}} = \overline{\sqrt{\mu_j^2 - 4 + \frac{4}{\alpha}}}. \tag{3.17}$$

From (3.9) we have that

$$\overline{\lambda_j^+} = \frac{\alpha}{2}\left(\overline{\mu_j} + \overline{\sqrt{\mu_j^2 - 4 + \frac{4}{\alpha}}}\right).$$

By using that $\overline{\mu_j} = \mu_i$ and (3.17) we get that

$$\overline{\lambda_j^+} = \frac{\alpha}{2}\left(\mu_i + \sqrt{\mu_i^2 - 4 + \frac{4}{\alpha}}\right) = \lambda_i^+.$$

Clearly the same proof can be repeated to prove that λ_i^- and λ_j^- are also complex conjugate. □

A situation in which the previous corollary applies is illustrated in Example 3.18 where μ_2 and μ_3 are complex conjugate and therefore λ_2^+ and λ_3^+ are also a complex conjugate pair (the same holds for λ_2^- and λ_3^-).

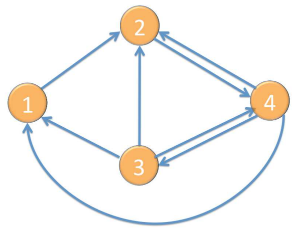

Figure 1: A graph with four nodes used in Example 3.18.

Example 3.18. *Given a four-noded graph whose adjacency matrix is*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

we have

$$P_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

and computing the spectrum of P_A *we get* $\{1, \frac{1}{3}(-1 \pm i\sqrt{2}), -\frac{1}{3}\}$*. Denoting*

$$\mu_2 = \frac{1}{3}(-1 + i\sqrt{2}), \quad \mu_3 = \frac{1}{3}(-1 - i\sqrt{2}), \quad \mu_4 = -\frac{1}{3}$$

by Theorem 3.8, the spectrum of M_A *is the union of* $\{1, 0\}$ *and the eigenvalues*

$$\begin{aligned} \lambda_2^+ &= 0.1967 + 0.1165i, \\ \lambda_2^- &= -0.4800 + 0.2842i, \\ \lambda_3^+ &= 0.1967 - 0.1165i, \\ \lambda_3^- &= -0.4800 - 0.2842i, \\ \lambda_4^+ &= 0.2425, \\ \lambda_4^- &= -0.5258, \end{aligned}$$

where we have used the typical value of $\alpha = 0.85$*.*

4. Spectrum of the matrix of the Multiplex PageRank

In this section we move to the Multiplex PageRank setting. While in the previous section we analyzed the relationships between $\sigma(M_A)$ and $\sigma(P_A)$, now it is natural to study the relationships between $\sigma(M_k)$ and $\sigma(\mathbb{B}_{1,1})$ for every $k \geq 2$.

First, note that if $\lambda \in \sigma(M_k)$, there exists a nonzero eigenvector $[\mathbf{u}^T\, \mathbf{w}^T]$ associated to λ, i.e., satisfying

$$[\mathbf{u}^T\, \mathbf{w}^T] M_k = \lambda [\mathbf{u}^T\, \mathbf{w}^T].$$

Hence

$$\left.\begin{array}{rcl} \frac{1}{k}\mathbf{u}^T \mathbb{B}_{1,1} + \alpha \mathbf{w}^T & = & \lambda \mathbf{u}^T \\ \frac{1-\alpha}{k}\mathbf{u}^T + \frac{1-\alpha}{k}\mathbf{w}^T \mathbf{e}[\mathbf{v}_1^T \ldots \mathbf{v}_k^T] & = & \lambda \mathbf{w}^T \end{array}\right\} \tag{4.1}$$

It is clear that $\lambda = 1$ is the dominant eigenvalue of M_k (note furthermore that M_k is row-stochastic, irreducible and primitive and therefore the eigenvalue 1 is simple and the unique with norm equal to the unity), while

$$\mu_d := \alpha + k - 1 \tag{4.2}$$

is the dominant eigenvalue of $\mathbb{B}_{1,1}$.

In this section $\mathbf{e}$ denotes the column vector of all ones in $\mathbb{R}^{k\,n \times 1}$.

We will see later that the eigenvectors $\mathbf{u}$ of $\mathbb{B}_{1,1}$ such that $\mathbf{u}^T\mathbf{e} = 0$ play a central role in the proof or the relationships obtained between $\sigma(M_k)$ and $\sigma(\mathbb{B}_{1,1})$ so we start this section by analyzing $\sigma(\mathbb{B}_{1,1})$ in terms of such eigenvectors.

Lemma 4.3. *The eigenvalues of $\mathbb{B}_{1,1}$ can be classified into two sets:*

(i) Those that have eigenvectors $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{e} \neq 0$. The only element of this set is μ_d.

(ii) Those that have eigenvectors $\mathbf{u}$ such that $\mathbf{u}^T\mathbf{e} = 0$. This happens for the rest of the eigenvectors of $\mathbb{B}_{1,1}$, including those associated to μ_d if it is multiple. If μ_d has multiplicity q, in this set we have μ_d (with multiplicity $q-1$) and the rest of eigenvalues μ_i.

Proof. The proof is analogous to the proof of Lemma 3.3. □

Now we will focus on the spectrum of M_k by starting with a series of technical lemmas.

Lemma 4.4. *Let $1 \neq \lambda \in \sigma(M_k)$ and let $[\mathbf{u}^T\ \mathbf{w}^T]$ be a (left) eigenvector associated to λ. Then, $\mathbf{u}^T\mathbf{e} + \mathbf{w}^T\mathbf{e} = 0$.*

Proof. Since $[\mathbf{u}^T\ \mathbf{w}^T]M_k = \lambda[\mathbf{u}^T\ \mathbf{w}^T]$, we have that

$$[\mathbf{u}^T\ \mathbf{w}^T] M_k \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix} = \lambda [\mathbf{u}^T\ \mathbf{w}^T] \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix} = \lambda(\mathbf{u}^T\mathbf{e} + \mathbf{w}^T\mathbf{e}), \tag{4.5}$$

but since $\lambda \neq 1$ and

$$M_k \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{e} \end{pmatrix}, \tag{4.6}$$

we get $(\mathbf{u}^T\mathbf{e} + \mathbf{w}^T\mathbf{e}) = 0$. □

Lemma 4.7. $\lambda = 0$ *is not an eigenvalue of* M_k.

Proof. Suppose on the contrary that $\lambda = 0 \in \sigma(M_k)$, i.e., $[\mathbf{u}^T\ \mathbf{w}^T]M_k = [\mathbf{0}, \mathbf{0}]$. Then, by (4.1) with $\lambda = 0$,

$$\left.\begin{array}{rcl} \frac{1}{k}\mathbf{u}^T\mathbb{B}_{1,1} + \alpha\mathbf{w}^T & = & \mathbf{0} \\ \frac{1-\alpha}{k}\mathbf{u}^T + \frac{1-\alpha}{k}\mathbf{w}^T\mathbf{e}[\mathbf{v}_1^T \ldots \mathbf{v}_k^T] & = & \mathbf{0} \end{array}\right\}$$

Hence, by multiplying the second equation by $\mathbf{e}$ on the right,

$$\frac{1-\alpha}{k}\mathbf{u}^T\mathbf{e} + \frac{1-\alpha}{k}\mathbf{w}^T\mathbf{e}k = 0,$$

leading to $\mathbf{u}^T\mathbf{e} = -k\mathbf{w}^T\mathbf{e}$. If $k \geq 2$ this contradicts $\mathbf{u}^T\mathbf{e} = -\mathbf{w}^T\mathbf{e}$ from the previous lemma. $\square$

Lemma 4.8. *If we take*

$$\widehat{\lambda} := \frac{1}{k}(k-1)(1-\alpha), \tag{4.9}$$

then $\widehat{\lambda} \in \sigma(M_k)$. *Moreover, if* $1 \neq \lambda \in \sigma(M_k)$ *is associated to an eigenvector* $[\mathbf{u}^T\,\mathbf{w}^T]$ *such that* $\mathbf{u}^T\mathbf{e} \neq \mathbf{0} \neq \mathbf{w}^T\mathbf{e}$, *then* $\lambda = \widehat{\lambda}$.

Proof. By using Lemma 4.3 we can consider an eigenvector $\mathbf{u}$ associated to μ_d such that $\mathbf{u}^T\mathbf{e} = 1$. Take

$$\mathbf{w}^T = \frac{1}{k-1}(\mathbf{u}^T - [\mathbf{v_1^T}\,\mathbf{v_2^T}\cdots\,\mathbf{v_k^T}]).$$

It is easy to see that

$$[\mathbf{u}^T\,\mathbf{w}^T]M_k = \widehat{\lambda}[\mathbf{u}^T\,\mathbf{w}^T].$$

Note that since $[\mathbf{v_1^T}\,\mathbf{v_2^T}\cdots\,\mathbf{v_k^T}]\mathbf{e} = k$ we obtain $\mathbf{w}^T\mathbf{e} = -1$.

On the other hand, suppose without loss of generality that $\mathbf{u}^T\mathbf{e} = -1$ and $\mathbf{w}^T\mathbf{e} = 1$. If we multiply the second equation of (4.1) by $\mathbf{e}$ on the right and use the fact that $\mathbf{u}^T\mathbf{e} = -1$ and $\mathbf{w}^T\mathbf{e} = 1$ we get

$$\lambda = \frac{1}{k}(k-1)(1-\alpha).$$

$\square$

Theorem 4.10. *Let* $\sigma(\mathbb{B}_{1,1}) = \{\mu_d\} \cup \{\mu_2, \cdots, \mu_m\}$, *then*

$$\sigma(M_k) = \{1, \widehat{\lambda}\} \cup \{\lambda_i^+, \lambda_i^-;\ \ 2 \leq i \leq m\},$$

where

$$\lambda_i^+ = \frac{\mu_i + \sqrt{\mu_i^2 + 4\alpha(1-\alpha)k}}{2k}, \quad \lambda_i^- = \frac{\mu_i - \sqrt{\mu_i^2 + 4\alpha(1-\alpha)k}}{2k}.$$

Proof. Let $\lambda \in \sigma(M_k)$ with an associated eigenvector $[\mathbf{u}^T\,\mathbf{w}^T]$ such that $\mathbf{u}^T\mathbf{e} = 0 = \mathbf{w}^T\mathbf{e}$. By using that $\mathbf{w}^T\mathbf{e} = 0$ in the second equation of (4.1) we get $\mathbf{w}^T = \frac{1-\alpha}{k\lambda}$. Now if we plug this in the first equation of (4.1) we get

$$\mathbf{u}^T\frac{1}{k}\mathbb{B}_{1,1} = \left(\lambda - \frac{\alpha(1-\alpha)}{k\lambda}\right)\mathbf{u}^T.$$

Therefore, we conclude that

$$\mu := \lambda k - \frac{\alpha(1-\alpha)}{\lambda} \tag{4.11}$$

is an eigenvalue of $\mathbb{B}_{1,1}$ associated to the eigenvector $\mathbf{u}^T$.

For the converse, let us study the eigenvalues of the matrix $\mathbb{B}_{1,1}$. Clearly

$$\mathbb{B}_{1,1}\begin{pmatrix}\mathbf{e}\\ \vdots\\ \mathbf{e}\end{pmatrix} = (\alpha+k-1)\begin{pmatrix}\mathbf{e}\\ \vdots\\ \mathbf{e}\end{pmatrix},$$

so $\mu = \alpha + k - 1$ is always an eigenvalue of $\mathbb{B}_{1,1}$. Our aim is to prove that the rest of the eigenvalues in the spectrum of $\mathbb{B}_{1,1}$ give rise to eigenvalues in the spectrum of M_k. We will also study the case when $\mu = \alpha + k - 1$ is a multiple eigenvalue of $\mathbb{B}_{1,1}$.

Now, let $\mu \in \sigma(\mathbb{B}_{1,1})$ such that $\mu \neq \alpha + k - 1$, or let $\mu = \alpha + k - 1$ be a multiple eigenvalue of $\mathbb{B}_{1,1}$. Let $\mathbf{u}^T$ be a (left) eigenvector associated to μ and such that $\mathbf{u}^T\mathbf{e} = 0$ (see Lemma 4.3).

As in the proof of Theorem 3.8, let us define the polynomial

$$m(x) = kx^2 - \mu x - \alpha(1-\alpha) \in \mathbb{R}[x],$$

and let λ be any of its roots. Notice that $\lambda \neq 0$ since $\alpha(1-\alpha) \neq 0$. Let us define

$$\mathbf{w}^T := \frac{1-\alpha}{\lambda k}\mathbf{u}^T.$$

Then it is easy to check that $[\mathbf{u}^T\,\mathbf{w}^T]$ satisfies (4.1), so the eigenvalue μ of $\mathbb{B}_{1,1}$ gives rise to the eigenvalues $\lambda^+, \lambda^- \in \sigma(M_k)$ as it was stated in the claim. □

Remark 4.12. The spectrum of M_k can be computed from the spectrum of $\mathbb{B}_{1,1}$. Apart from the eigenvalues 1 and $\frac{1}{k}(k-1)(1-\alpha)$, which always belong to $\sigma(M_A)$, for any $\mu \neq \alpha + k - 1$ we obtain two eigenvalues

$$\lambda^+ = \frac{\mu + \sqrt{\mu^2 + 4\alpha(1-\alpha)k}}{2k}, \quad \lambda^- = \frac{\mu - \sqrt{\mu^2 + 4\alpha(1-\alpha)k}}{2k} \in \sigma(M_k).$$

Furthermore, if $\alpha + k - 1$ is a multiple eigenvalue of $\mathbb{B}_{1,1}$ also

$$\lambda^+ = \frac{\alpha + k - 1 + \sqrt{(\alpha+k-1)^2 + 4\alpha(1-\alpha)k}}{2k} \in \sigma(M_k),$$

$$\lambda^- = \frac{\alpha + k - 1 - \sqrt{(\alpha+k-1)^2 + 4\alpha(1-\alpha)k}}{2k} \in \sigma(M_k).$$

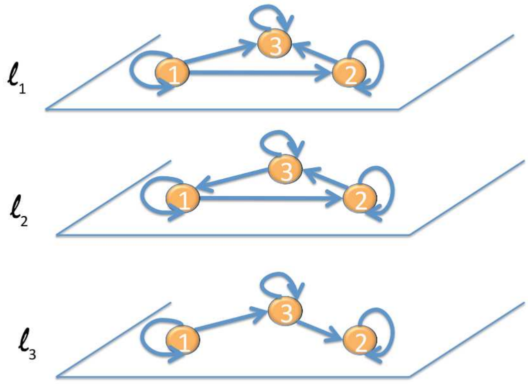

Figure 2: A multiplex example with 3 layers and 3 nodes used in Example 4.15.

As in the previous Section, note that Theorem 4.10 allows computing the spectrum of a $2kn \times 2kn$ matrix (M_k) in terms of the spectrum of a smaller $kn \times kn$ matrix ($\mathbb{B}_{1,1}$), so the computational complexity decreases its order.

Corollary 4.13. *Given the eigenvalues λ_i^+ and λ_i^-, then for every $i = 2, \dots m$.*

$$\lambda_i^+ \lambda_i^- = \frac{1}{k}\alpha(\alpha - 1) < 0.$$

Corollary 4.14. *Given the eigenvalues λ_i^+ and λ_i^-, if μ_i and μ_j are complex conjugate eigenvalues of $\mathbb{B}_{1,1}$ then λ_i^+ and λ_j^+ are complex conjugate eigenvalues of M_k. The same holds for λ_i^- and λ_j^-.*

Example 4.15. *Let $k = 3$, $\alpha = \frac{3}{4}$ and take the multiplex network whose adjacency matrices are given by*

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\mathbb{B}_{1,1} = \left(\begin{array}{ccc|ccc|ccc} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{8} & \frac{3}{8} & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & \frac{3}{4} & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & \frac{3}{8} & 0 & \frac{3}{8} \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \frac{3}{4} \end{array}\right).$$

The spectrum of $\mathbb{B}_{1,1}$ *is*

$$\sigma(\mathbb{B}_{1,1}) = \left\{ \begin{array}{lcl} \mu_1 & = & \frac{11}{4} \\ \mu_2 & = & \frac{1}{16}\left(13+\sqrt{593+32i\sqrt{2}}\right) \\ \mu_3 & = & \frac{1}{16}\left(13+\sqrt{593-32i\sqrt{2}}\right) \\ \mu_4 & = & \frac{1}{16}\left(13-\sqrt{593-32i\sqrt{2}}\right) \\ \mu_5 & = & \frac{1}{16}\left(13-\sqrt{593+32i\sqrt{2}}\right) \\ \mu_6 & = & -\frac{5}{8} \quad (double) \\ \mu_7 & = & -\frac{1}{4} \quad (double) \end{array} \right\}$$

The spectral radius is $\mu_d = \alpha + k - 1 = \mu_1$ *that is simple. In this example we have* $\widehat{\lambda} = \frac{1}{k}(k-1)(1-\alpha) = \frac{1}{6}$. *By taking the personalization vectors* $\mathbf{v_1^T} = \mathbf{v_2^T} = \mathbf{v_3^T} = \frac{1}{3}(1,1,1)$ *we obtain:*

$$M_3 = \frac{1}{3}\begin{pmatrix} \mathbb{B}_{1,1} & \mathbb{B}_{1,2} \\ \mathbb{B}_{2,1} & \mathbb{B}_{2,2} \end{pmatrix}$$

where $\mathbb{B}_{1,1}$ *has been given above,* $\mathbb{B}_{1,2} = \frac{1}{4}I_9$, $\mathbb{B}_{2,1} = \frac{9}{4}I_9$, *and*

$$\mathbb{B}_{2,2} = \frac{1}{12}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The computation of the eigenvalues of M_3 *gives*

$$\sigma(M_3) = \left\{ \begin{array}{lcl} \lambda_1 & = & 1.0000 \\ \lambda_2 & = & 0.8519+0.0178i \\ \lambda_3 & = & 0.8519-0.0178i \\ \lambda_4 & = & -0.3949+0.0138i \\ \lambda_5 & = & -0.3949-0.0138i \\ \lambda_6 & = & -0.0733+0.0015i \\ \lambda_7 & = & -0.0733-0.0015i \\ \lambda_8 & = & 0.1581+0.0055i \\ \lambda_9 & = & 0.1581-0.0055i \\ \lambda_{10} & = & -\frac{3}{8} \quad (double) \\ \lambda_{11} & = & -\frac{1}{24}\left(1+\sqrt{37}\right) \quad (double) \\ \lambda_{12} & = & -\frac{1}{24}\left(1-\sqrt{37}\right) \quad (double) \\ \lambda_{13} & = & \frac{1}{6} \quad (triple) \end{array} \right\}.$$

Let us analyze the theoretical form of the spectrum of M_3 *to illustrate Theorem 4.10.* $\lambda_1 = 1$ *is the expected spectral radius. One of the values of* λ_{13}

corresponds to $\widehat{\lambda}$. The rest of eigenvalues derive from μ_i according to the following:

λ_2 *comes from*	μ_2	*by using the formula of*	λ_2^+
λ_3 *comes from*	μ_3	*by using the formula of*	λ_3^+
λ_4 *comes from*	μ_4	*by using the formula of*	λ_4^+
λ_5 *comes from*	μ_5	*by using the formula of*	λ_5^+
λ_6 *comes from*	μ_2	*by using the formula of*	λ_2^-
λ_7 *comes from*	μ_3	*by using the formula of*	λ_3^-
λ_8 *comes from*	μ_4	*by using the formula of*	λ_4^-
λ_9 *comes from*	μ_5	*by using the formula of*	λ_5^-
Both λ_{10} *come from*	μ_6	*by using the formula of*	λ_6^+
λ_{11} *comes from*	μ_7	*by using the formula of*	λ_7^+
λ_{12} *comes from*	μ_7	*by using the formula of*	λ_7^-
Two values of λ_{13} *come from*	μ_6	*by using the formula of*	λ_6^-

Note also that this example illustrates Corollary 4.13 and. Corollary 4.14.

5. Conclusions and final remarks

We have related the spectrum of the matrices associated with the computation of the *Multiplex PageRank*, both in the monoplex case and in the multiplex cases, which allows reducing the computational complexity of this problem. In particular, if $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a (monoplex) network with n nodes and whose associated two-layer approach stochastic $2n \times 2n$ matrix is given by M_A, then Theorem 3.8 shows that $\sigma(M_A)$ can be computed in terms of $\sigma(P_A)$, where P_A is the row stochastic $n \times n$ matrix associated to $\mathcal{G}$. In the multiplex setting, Theorem 4.10 proves the connection between $\sigma(M_k)$ and $\sigma(\mathbb{B}_{1,1})$, where M_k is the $2kn \times 2kn$ matrix giving the Multiplex PageRank of a multiplex network and $\mathbb{B}_{1,1}$ is the $kn \times kn$ matrix containing the adjacency information of the corresponding layers.

It is also worth noting that some of the obtained results show that the concept of two-layer approach PageRank has some traits similar to the concept of classic PageRank. In particular, we show that the spectrum does not depend on the personalization vector (Remark 3.12) and that if we sum two by two the eigenvalues of the matrix M_A associated to the two-layer approach PageRank, we obtain the eigenvalues of the matrix G associated to the classic PageRank (Remark 3.14).

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