On the α-nonbacktracking centrality for complex networks: existence and limit cases

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Abstract

Non-bactracking centrality was introduced as an attempt to correct a deficiency of eigenvector centrality, since eigenvector centrality in a network can be artificially increased in nodes with high degree (hubs) because a hub is central because its neighbors are central, but these in turn are central only by the hub. In this work we introduce the α-nonbacktracking centrality as an extension to solve some problems related to the uniqueness of non-bactracking principal eigenvector. This extension makes it possible to demonstrate the convergence of the α-centrality principal eigenvector when $\alpha \to 0$ and also the convergence of PageRank vectors when the damping factor tends to 1, which gives an idea of the applicability of this new measure of centrality.

Keywords: Non-backtracking centrality, alpha-centrality, perturbative analysis of matrices, spectral analysis of complex networks

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1. Introduction

Complex networks have been used with great success to model many real-world systems in fields ranging from biology (which include issues such as metabolic pathways, protein folding or genetic regulatory networks) to the Internet, the World Wide Web and other technological systems [2, 3, 11, 12, 15, 17, 18, 23]. Research on these issues must necessarily encompass a diversity of views that include different complementary aspects of the network structure,

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and the huge complexity of these objects requires of new tools coming from other fields, including matrix analysis, statistical mechanics and computer sciences [1, 2, 4, 3, 11, 26]. So, complex networks have recently attracted the attention of numerous research areas and consequently the mathematical and computational study of complex networks has experienced very significant growth in recent years [3, 12, 23]. The spectral properties of the adjacency matrix provide great insight into the structure and function of complex networks. Specifically, the largest eigenvalue (spectral radius) and its associated principal eigenvector are fundamental in the understanding of nodes' centrality, since the number of connections a node has to other nodes (its degree) is a measure of centrality for quantifying the importance of a node, but it's easy to see that not all edges are the same when you want to quantify the influence or importance of a particular node. Eigenvector centrality [5] takes into account that not all the edges or relations are equal, since a node is more important (or influential) if the nodes with which it is connected are, in turn, important or influential nodes. Eigenvector Centrality has been analyzed and extended for use in different contexts and applications [3, 10, 21, 24, 25, 26]. Specifically, in [21] the authors very rightly observe that eigenvector centrality in a network can be artificially high on nodes with high degree (hubs). The reason is simple and can be easily grasped in the following sentence:

I am important, then my friends become important, then I become even more important, then...

Thus the pattern "My importance depends on my friend's importance" tends to give priority to hubs just for their own nature, i.e., a hub with an elevated eigenvector centrality transmits it to its neighbors, who in turn reflect it again and inflate the hub's centrality artificially. Therefore, if we can avoid this reflection, centrality will behave much more realistically. Thus, in [21] an attempt to correct this weakness is proposed by using the non-backtracking centrality. The idea is to use a modified eigenvector centrality that is similar in many ways but with an important change: to calculate the centrality of a particular node, the authors consider the centrality of its neighbors, in a similar way as is done with the usual centrality eigenvector, but the centrality of its neighbors is now calculated in the absence of that particular node. This centrality measure can be calculated using the Hashimoto or non-backtracking matrix [14, 20], as pointed out by the authors in [21]. As we will see in the next section, this matrix is closely related to the adjacency matrix of the line graph corresponding to the network under consideration. In any case, it is important to highlight that when the authors in [21] introduced the non-backtracking centrality vector a natural objection arises: can we be sure that such a vector exists? More precisely, Perron's theorem ([22]) guarantees that a non-negative and irreducible square matrix A has a non-negative eigenvector associated to its spectral radius $\rho(A)$. But if A were not irreducible then Perron-Frobenius theorem can no longer be used and hence the unicity of eigenvector gets compromised; but, however it is this unicity which allows us to speak of *the* nonbacktracking centrality eigenvector.

The main goal of this work is to define of a new centrality measure, the α-

nonbacktracking centrality, that makes possible to solve the problems related to the uniqueness of non-bactracking principal eigenvector. This extension makes possible to demonstrate the convergence of the α-centrality principal eigenvector when $\alpha \to 0$ and also the convergence of PageRank vectors when the damping factor tends to 1, suggesting new ideas on the applicability of this new measure of centrality

Throughout the following sections we will develop these ideas and results, and we will see how it is possible to guarantee the uniqueness of this eigenvector.

The structure of the paper is as follows. After this introduction, Section 2 is devoted to introduce and recall some preliminary results and definitions. In Section 3 the α-nonbacktracking centrality is introduced in order to avoid the problem of irreducibility and lack of uniqueness of the non-backtracking principal eigenvector. Finally, in Section 4 a result related to the convergence of the spectral radii and the Perron vectors of a family of irreducible non-negative matrices is obtained and, moreover, we point out how our result can be used to prove the convergence of the α-centrality vectors when $\alpha \to 0$ and the PageRank vectors when the damping factor tends to 1.

2. Notation and preliminaries

Although the framework in which we will develop the results is that of non-directed graphs with no loops, in this section we consider a non directed graph $G = (V, E)$ with no loops and $|V| = n$, $|E| = m$, the cardinal of nodes and edges. If $i \to j$ is the one-way trip from i to j then $j \to i$ is the return trip from j to i.

Consider

$$\overrightarrow{E} \doteqdot \{(i,j),(j,i);\{i,j\} \in E\} = \{i \to j, j \to i; \{i,j\} \in E\}$$

the edge-set obtained from E by adding all the return trips. Clearly $|\overrightarrow{E}| = 2m$

Thus, in order to remove feedback the heuristics goes as follows: the centrality of edge $k \to l$ is proportional to the sum of the centralities of all edges **incident** on $k \to l$ **except** edge $l \to k$. Here $i \to j$ is incident on $k \to l$ if $j = k$.

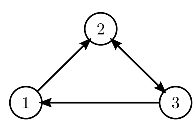

Figure 1: An example of a directed graph G with 3 nodes to obtain $L(G)$ and the graph associated to $B_1(G)$.

The heuristics is revealed in the Hashimoto matrix ([14]) as follows: We start fixing an order in $\overrightarrow{E}$ (for instance the lexicographic order), then we take the adjacency matrix for edge incidence

$$(B_1)_{i\to j,k\to l} = \begin{cases} 1 & j = k \text{ and } i \neq l \\ 0 & \text{otherwise} \end{cases}$$

that is,

$$(B_1)_{i\to j,k\to l} = \delta_{jk}(1-\delta_{il}),$$

where δ_{ij} is the Kronecker's delta, i.e.,

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases} .$$

Notice that the Hashimoto matrix is closely related to the adjacency matrix of $\overrightarrow{L}(G)$, the linegraph of G

$$(M_{\overrightarrow{L}(G)})_{i\to j,k\to l} = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & \text{otherwise} \end{cases}$$

(here $i \to j$ is incident on $k \to l$ if $j = k$).

Thus, in the example proposed for the graph G considered, by taking the lexicographic order on edges: $\{(1,2);(2,3);(3,1);(3,2)\}$.

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{\overrightarrow{L}(G)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Notice that the Hashimoto matrix is B_1 and is "part" of the adjacency matrix $M_{\overrightarrow{L}(G)}$ of the line graph of $\overrightarrow{G} = (V, \overrightarrow{E})$

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{\overrightarrow{L}(G)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The relation between B_1, the adjacency matrix of $L(G)$ and their respective associated graphs is illustrated in Figures 1, 2 and 3.

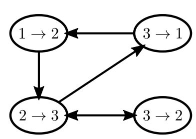

Figure 2: The line graph $L(G)$

Remark 2.1.

$$\rho(B_1) \leq \rho(\overrightarrow{L}(G))$$

It is important to highlight the existence of strong relationships between the eigenvector centrality of a given graph G, the eigenvector centrality of its linegraph $L(G)$ linegraph and the eigenvector centrality of the bipartite graph

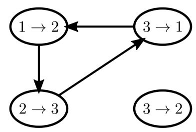

Figure 3: The graph associated to B_1

$B(G)$ associated to G, both in the case of G being an undirected network [8] and in the case of G being a directed network [9].

Thus the non-backtracking centrality of edges is a function $c_1 : \overrightarrow{E} \to [0,1]$ satisfies:

- $\sum_{k\to l\in\overrightarrow{E}} c_1(k \to l) = 1$ (normalization).
- $c_1(k \to l)$ is proportional to the sum of $c_1(j \to k)$ where $j = l$ is disregarded.

In terms of B_1

$$c_1(i \to j) \equiv c_{1\,i\to j} = \frac{1}{\lambda} \sum_{k\to l\in\overrightarrow{E}} B_{1\,k\to l, i\to j} c_{1\,k\to l}$$

Thus, if $c_1 = ((c_1)_{i\to j})_{i\to j\in\overrightarrow{E}}$ is a column vector, then $\lambda c_1 = B_1^t c_1$

We remark that the non-backtracking centrality of each edge in $\overrightarrow{E}$ is ranked by means of a normalized non-negative eigenvector of B_1^t.

At this point, once the non-backtracking centrality of edges has been defined ,it is possible to define the non-backtracking centrality of nodes in the following natural way

Definition 2.2. The **centrality of node** i is the sum of centralities of all edges incident on i, in other words, the sum of centralities of all edges $k \to i$.

3. α-nonbacktracking centrality: existence and computation

In this section the α-nonbacktracking centrality is introduced. The objective is twofold:

a) First and foremost we want to avoid the irreducibility problem that results in lack of unicity for the nonbacktracking eigenvector.

b) Secondly we want to unify the different centralities into a single definition that interpolates between non-backtracking centrality and eigenvector centrality.

We pass to propose the heuristics that support a definition consistent with our goal. Given an order in E and $\alpha \in [0,1]$, the edge α-nonbacktracking centrality $\mathbf{c}(\alpha) : E \to [0,1]$ should satisfy:

1) $\sum_{k\to l\in E} c_{k\to l}(\alpha) = 1$ (normalization).

2) $c_{k\to l}(\alpha)$ is proportional to the sum of $c_{j\to k}(\alpha)$, where $j \to k$ is incident on $k \to l$ and instance $j = l$ is admissible although dampened by α.

Thus, the edge incidence information is collected in the following edge adjacency matrix

$$B_{i\to j,k\to l}(\alpha) = \begin{cases} 1 & j = k \text{ and } i \neq l \\ \alpha & j = k \text{ and } i = l \\ 0 & \text{otherwise} \end{cases} ;$$

that is,

$$B_{i\to j,k\to l}(\alpha) = \delta_{jk}(1 + (\alpha - 1)\delta_{il})\,.$$

Remark 3.1. For $\alpha = 0$ we get the Hashimoto matrix $B(0)$, while for $\alpha = 1$ the matrix of $\overrightarrow{L}(G)$ is recovered.

Theorem 3.2. *Let G be a directed and connected network. Then, G is strongly connected if and only if $B(\alpha)$ is irreducible for $\alpha \in (0, 1]$.*

Proof. It is well known that G is strongly connected if and only if $\overrightarrow{L}(G)$ is strongly connected (cf. [27], p.44), or equivalently $B(1)$ is irreducible. Analogously, G is strongly connected if and only if $B(\alpha)$ is irreducible for $\alpha \in (0, 1]$. □

In the following, we suppose that G be a directed and strongly connected network.

Let $\alpha \in (0, 1]$. We define the edge α-nonbacktracking centrality $\mathbf{c}(\alpha) = ((c_{i\to j}(\alpha))_{i\to j\in E}$ as the positive normalized eigenvector associated to the spectral radius of $B^t(\alpha)$.

Definition 3.3. Let $\alpha \in (0, 1]$. The α-nonbacktracking centrality $c_i(\alpha)$ of node i is the sum of centralities $c_{i\to k}(\alpha)$ of the edges $i \to k$ (edges incident on i).

Remark 3.4. In the next section, we will prove that $\lim_{\alpha\to 0} \mathbf{c}(\alpha)$ exists. Call this limit c. Evidently c is non negative and has norm one. Also c is easily seen to belong to the eigenspace associated to the spectral radius $\rho(B(0))$. Note that, due to the possible lack of irreducibility of $B(0)$, such eigenspace may have a dimension greater than 1; hence choosing one centrality eigenvector in this eigenspace is certainly ambiguous. In this context c clearly appears as a natural candidate as *the* non-backtracking centrality eigenvector of $B(0)$.

Now, we are interested in computing $c_i(\alpha)$ and $\rho(B(\alpha))$ with the help of matrix smaller than $B(\alpha)$. We consider the non-directed case.

Fix an order in E. For a given $v = (v_{i\to j})_{i\to j\in E} \in \mathbb{R}^m$ and a given node $i \in V$ we denote

$$v_i^{\text{out}} = \sum_{j\in N(i)} v_{i\to j} = \sum_{j=1}^{n} a_{ij} v_{i\to j}, \quad v_i^{\text{in}} = \sum_{j\in N(i)} v_{j\to i} = \sum_{j=1}^{n} a_{ji} v_{j\to i}\,,$$

where we assume $v_{i\to j} = 0$ when $i \to j \notin E$. Now we define

$$v^{\text{out}} = (v_i^{\text{out}})_{i=1}^n,\ v^{\text{in}} = (v_i^{\text{in}})_{i=1}^n\,.$$

Notice that, with this notation,

$$c_i(\alpha) = \sum_{j\in N(i)} c_{j\to i}(\alpha) = \sum_{j=1}^{n} a_{ji} c_{j\to i}(\alpha) = c_i^{\text{in}}(\alpha)\,.$$

Theorem 3.5. *Let G be a non directed graph with no loops and let $\alpha \in (0,1]$. If $v \in \mathbb{R}^m$ is an eigenvector of $B^t(\alpha)$, with eigenvalue λ, and either $v^{out} \neq 0$ or $v^{in} \neq 0$, then $\begin{pmatrix} v^{out} \\ v^{in} \end{pmatrix}$ is an eigenvector of $\tilde{B}(\alpha)$, with the same eigenvalue λ, with $\tilde{B}(\alpha) \equiv \begin{pmatrix} 0 & Gr + (\alpha-1)I_n \\ (\alpha-1)I_n & A \end{pmatrix}$, where Gr is the diagonal matrix which elements are the degrees $d(i)$. In particular, if $v > 0$ is an eigenvector corresponding to to the spectral radius $\rho(B(\alpha))$, then*

$$\rho(B(\alpha)) \begin{pmatrix} v^{out} \\ v^{in} \end{pmatrix} = \tilde{B}(\alpha) \begin{pmatrix} v^{out} \\ v^{in} \end{pmatrix}.$$

Proof. Let $v \in \mathbb{R}^m$ and $k \to l \in E$. Then

$$\begin{aligned}
(B^t(\alpha)\, v)_{k\to l} &= \sum_{x\to y\in E} B^t_{k\to l, x\to y}(\alpha)\, v_{x\to y} = \sum_{x\to y\in E} B_{x\to y, k\to l}(\alpha)\, v_{x\to y} \\
&= \sum_{x,y=1}^{n} \delta_{yk}(1 + (\alpha-1)\delta_{xl}) a_{xy}\, v_{x\to y} = \alpha\, a_{lk}\, v_{l\to k} + \sum_{x\neq l} a_{xk} v_{x\to k} \\
&= (\sum_{x=1}^{n} a_{xk}\, v_{x\to k}) + (\alpha-1) a_{lk}\, v_{l\to k}\,.
\end{aligned}$$

Therefore, for $i \in V$

$$\begin{aligned}
(B^t(\alpha)\, v)_i^{\text{out}} &= \sum_{j=1}^{n} a_{ij} (B^t(\alpha)\, v)_{i\to j} = \sum_{j=1}^{n} a_{ij} \left((\sum_{x=1}^{n} a_{xi}\, v_{x\to i}) + (\alpha-1) a_{ji}\, v_{j\to i} \right) \\
&= \left(\sum_{j\in N(i)} \sum_{x\in N(i)} v_{x\to i} \right) + (\alpha-1) \sum_{j\in N(i)} v_{j\to i} = |N(i)| v_i^{in} + (\alpha-1) v_i^{in} \\
&= (d(i) + (\alpha-1)) v_i^{in}\,,
\end{aligned}$$

where $d(i)$ is the degree of node i. Analogously, for $i \in V$

$$(B^t(\alpha)\, v)_i^{\text{in}} = \sum_{j=1}^n a_{ji}(B^t(\alpha)\, v)_{j\to i} = \sum_{j=1}^n a_{ji} \left((\sum_{x=1}^n a_{xj}\, v_{x\to j}) + (\alpha-1) a_{ij}\, v_{i\to j} \right)$$

$$= \left(\sum_{j\in N(i)} \sum_{x\in N(j)} v_{x\to j} \right) + (\alpha-1) \sum_{j\in N(i)} v_{i\to j} = \sum_{j=1}^n a_{ji} v_j^{in} + (\alpha-1) v_i^{out}\,.$$

It follows

$$\begin{pmatrix} (B^t(\alpha)\, v)^{\text{out}} \\ (B^t(\alpha)\, v)^{\text{in}} \end{pmatrix} = \tilde{B}(\alpha) \begin{pmatrix} v^{\text{out}} \\ v^{\text{in}} \end{pmatrix}.$$

Then, if $v \in \mathbb{R}^m$ is an eigenvector of $B^t(\alpha)$, with eigenvalue λ, we have

$$(B^t(\alpha)\, v)_i^{\text{out}} = \sum_{j=1}^n a_{ij}(B^t(\alpha)\, v)_{i\to j} = \lambda \sum_{j=1}^n a_{ij} v_{i\to j} = \lambda v_i^{\text{out}}$$

and, analogously,

$$(B^t(\alpha)\, v)_i^{\text{in}} = \lambda v_i^{\text{in}}\,.$$

Therefore

$$\lambda \begin{pmatrix} v^{\text{out}} \\ v^{\text{in}} \end{pmatrix} = \tilde{B}(\alpha) \begin{pmatrix} v^{\text{out}} \\ v^{\text{in}} \end{pmatrix}.$$

□

Remark 3.6. $\tilde{B}(1) = \begin{pmatrix} 0 & Gr \\ 0 & A \end{pmatrix}$. Thus

$$\sigma(\tilde{B}(\alpha)) = \sigma(A) \cup \{0\}$$

Positive $\tilde{B}(1)$-eigenvectors cannot have 0 as eigenvalue, thus the 1-nonbacktracking centrality of nodes coincides with the eigenvector centrality.

4. Limit case of α-nonbacktracking centrality and other spectral centrality measures

In this section we will study the convergence when $\alpha \to 0$ of the spectral radii and the Perron vectors of a family of irreducible non-negative matrices M_α of the form $M_0 + \alpha M$. This is the case when $M_0 = B(0)^t$ is the transpose of the Hashimoto matrix perturbed by an α-multiple of a permutation matrix M, as proposed in Section 3. We will also point out how our result can be used to prove the convergence of the α-centrality vectors when $\alpha \to 0$ and the PageRank vectors when the damping factor tends to 1.

Clearly $M_0 + \alpha M$ converges to M_0 as $\alpha \to 0$ and therefore $\rho(M_0 + \alpha M) \to \rho(M_0)$. This is well known and uses the fact that the composition of continuous functions is continuous. Indeed, the coefficients of a matrix depend continuously on the matrix (whatever the matrix norm considered) and therefore the characteristic polynomial depends continuously on the matrix. As the roots of the characteristic polynomial (over $\mathbb{C}$) continuously depend on its coefficients (see, for example [13]) and the absolute value is a continuous function we obtain, by composing all four previous dependencies, that the spectral radius varies continuously with the matrix.

Next we will justify that the family of Perron vectors of M_α converges as α goes to zero. The techniques used are known and make use of well established facts on analytical functions ([16, 19]). We pass to collect the details in our setting.

Theorem 4.1. *Let $M_\alpha = M_0 + \alpha M$, $\alpha > 0$, be a family of irreducible non-negative matrices. Let $\mathbf{c}(\alpha)$ be the Perron vector of each M_α, $\alpha > 0$. Then the sequence of vectors $\{\mathbf{c}(\alpha)\}_\alpha$ converges when $\alpha \to 0$.*

Proof. Let ρ_α be the spectral radius of M_α and let $\mathbf{c}(\alpha) = (c_1(\alpha), \ldots, c_n(\alpha))$ be the Perron vector of M_α. Let $A(\alpha) := M_\alpha - \rho_\alpha I$ where I denotes the identity matrix I of size $n \times n$.

• Step 1: Let us express each $c_j(\alpha)$, $j = 1, \ldots, n$, in terms of adjoints of elements of the matrix $A(\alpha)$. Since M_α is irreducible and ρ_α is the spectral radius, $rg(A(\alpha)) = n-1$. Let $\tilde{A}(\alpha)$ be the submatrix of $A(\alpha)$ containing the first $n-1$ rows. Without loss of generality we can assume that the rank of $\tilde{A}(\alpha)$ is $n-1$ and that the following linear map F is bijective:

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{A}(\alpha) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ x_n \end{pmatrix}.$$

The matrix of this linear map with respect to the canonical basis will be denoted F again. Notice that the matrix F coincides with the matrix $A(\alpha)$ where the last row was replaced by $(0, 0, \ldots, 0, 1)$. Thus, the minors of the elements of the last row are the same for both matrices. Consequently $\det(F) = A(\alpha)_{nn}$, where $A(\alpha)_{nn}$ is the minor of the element a_{nn} of the matrix $A(\alpha)$. In general, let us denote $A(\alpha)_{ij}$ the minor of the element a_{ij} of $A(\alpha)$.

Since $F(\mathbf{c}(\alpha)^t) = (\vec{0}_{\mathbb{R}^{n-1}}, c_n(\alpha))^t$ and F is bijective,

$$\mathbf{c}(\alpha)^t = F^{-1} \begin{pmatrix} \vec{0}_{\mathbb{R}^{n-1}} \\ c_n(\alpha) \end{pmatrix} = c_n(\alpha)(F^{-1})_n,$$

where $(F^{-1})_n$ denotes the n^{th} column of F^{-1}. From classic Cramer's rule we get that

$$\mathbf{c}(\alpha)^t = c_n(\alpha)(F^{-1})_n = \frac{c_n(\alpha)}{\det(F)} \begin{pmatrix} (-1)^{n+1}A(\alpha)_{n1} \\ (-1)^{n+2}A(\alpha)_{n2} \\ \dots \\ (-1)^{2n}A(\alpha)_{nn} \end{pmatrix}$$
$$= \frac{c_n(\alpha)}{A(\alpha)_{nn}} \begin{pmatrix} (-1)^{n+1}A(\alpha)_{n1} \\ (-1)^{n+2}A(\alpha)_{n2} \\ \dots \\ (-1)^{2n}A(\alpha)_{nn} \end{pmatrix}$$

and thus

$$c_j(\alpha) = c_n(\alpha)\frac{(-1)^{n+j}A(\alpha)_{nj}}{A(\alpha)_{nn}}, \quad j = 1, \dots, n.$$

Notice that $c_j(\alpha) > 0$ implies $A(\alpha)_{nj} \neq 0$ for every j. Moreover, since $\mathbf{c}(\alpha)$ is the Perron vector,

$$1 = \|\mathbf{c}(\alpha)\|_1 = \frac{c_n(\alpha)}{|A(\alpha)_{nn}|} \sum_j |A(\alpha)_{nj}|,$$

which implies that $|A(\alpha)_{nn}| = c_n(\alpha) \sum_j |A(\alpha)_{nj}|$ and therefore

$$c_j(\alpha) = \frac{|A(\alpha)_{nj}|}{\sum_i |A(\alpha)_{ni}|} = \frac{1}{1 + \sum_{i\neq j} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}}. \tag{4.2}$$

Observe that each term $\frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}$ in (4.2) is a quotient of polynomials in two variables α and $\rho_\alpha = \rho(\alpha)$.

• Step 2: Let us prove that for every $j \in \{1, \dots, n\}$ and every $i \neq j$, the limit

$$\lim_{\alpha\downarrow 0} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}$$

always exists and belongs to $[0, \infty]$. Take the function $G(\alpha, \omega) = \det(M_\alpha - \omega I)$ defined on $\mathbb{C} \times \mathbb{C}$. Notice that $G(\alpha, \omega)$ is a polynomial of the two variables α, ω of degree n. Hence

$$G(\alpha, \omega) = \sum_{k=0}^{n} g_k(\alpha)\omega^k,$$

where $g_k(\alpha)$ is a polynomial in α.

By choosing, for every $\alpha \in \mathbb{C}$, an element ω_α in the spectrum $\sigma(M_\alpha)$ of M_α we have $G(\alpha, \omega_\alpha) = 0$. Notice that for every $\alpha \in \mathbb{R}^+ \cup \{0\}$ we can choose $\omega_\alpha = \rho_\alpha$, the spectral radius of M_α (by the Perron-Frobenius theorem for nonnegative matrices and the continuity of the spectral radius observed above).

Thus, we have a function $\omega(\alpha) = \omega_\alpha$ defined on the complex plane satisfying $\omega(\alpha) = \rho_\alpha$ for $\alpha \in \mathbb{R}^+ \cup \{0\}$ and such that $G(\alpha, \omega_\alpha) = 0$ for all $\alpha \in \mathbb{C}$. According to §13 in [19] such a function, $\omega(\alpha)$, is called an element of an *algebraic function* defined by $G(\alpha, \omega) = 0$.

We can assume that equation $G(\alpha, \omega) = 0$ is irreducible (see [19], section §13). Observe that for a fixed α_0 the equation $G(\alpha_0, \omega) = 0$ will have, in general, n distinct roots $\omega_0^{(i)}$, $i = 1, \ldots, n$. As observed in§13 in [19], there are two exceptions to this:

(i) $g_n(\alpha_0) = 0$. Then the degree (in ω) of the equation $G(\alpha_0, \omega) = 0$ is lowered.

(ii) $G(\alpha_0, \omega) = 0$ has multiple roots.

As noticed in Section §13 of [19], there are only finitely many values of α, which we denote $a_1, \ldots, a_r$, for which either *(i)* or *(ii)* holds. These special values are called *critical points* and must therefore be isolated points. This implies that when the limit $\lim_{\alpha \downarrow 0} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}$ is considered (here $\alpha \downarrow 0$ in the positive real axis), there must exist some $\delta > 0$ such that every $\alpha \in (0, \delta)$ is a non-critical point, while $\alpha = 0$ can be a critical point.

If α_0 is not one of the critical points above then in Section §14 of [19] the equation $G(\alpha, \omega) = 0$ is proved to define a single n-valued analytical function $\omega = F(\alpha)$. Equivalently, this equation defines n-branches of an multivalued analytical function. More importantly the critical points become poles of this function as proved in Section §15 of [19].

Since $\rho(\alpha)$ satisfies $G(\alpha, \rho(\alpha)) = 0$ for all $\alpha \in [0, \delta)$ it follows from the above that $\rho(\alpha)$ is an analytical function with 0 as a pole. But this certainly means that after applying L'Hopital's rule at most a finite number of times we get

$$\lim_{\alpha \downarrow 0} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|} \in [0, \infty].$$

• Step 3: The number of terms in the sum $\sum_{i \neq j} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}$ is finite. Hence from Step 2 it follows that

$$c_j(\alpha) = \frac{1}{1 + \sum_{i \neq j} \frac{|A(\alpha)_{ni}|}{|A(\alpha)_{nj}|}}$$

must converge and therefore the vector $\mathbf{c}(\alpha) = (c_1(\alpha), \ldots, c_n(\alpha))$ must converge.

□

As was observed above, Theorem 4.1 allows us to give a precise meaning to the nonbacktracking centrality eigenvector of a network G even if the associated nonbacktracking matrix, $B(0)$, is not irreducible. Indeed, in this case the eigenspace associated to the spectral radius may have dimension greater than one and it could be not possible to select a unique norm-one positive vector inside. But for every α close to zero we can find the α-nonbacktracking centrality eigenvector $\mathbf{c}(\alpha)$ of $M_\alpha := B(\alpha)^t$ and take $\mathbf{c} = \lim_{\alpha \downarrow 0} \mathbf{c}(\alpha)$ (which has been shown to exist) as α decreases to zero. It is straightforward to check that

$\mathbf{c}$ has norm one, is non-negative and belongs to the eigenspace associated to the spectral radius of $M_0 := B(0)^t$. Therefore it is perfectly sound to refer to $\mathbf{c}$ as the nonbacktracking centrality eigenvector of G. The following definition collects this remark.

Definition 4.3. Given a directed network G we call the *nonbacktracking edge centrality* of G to the vector $\mathbf{c} = \lim_{\alpha\downarrow 0} \mathbf{c}(\alpha)$. From this we define the nonbacktracking centrality of a node $i \in G$ as

$$c_i = \sum_j c_{i\to j},$$

where $c_{i\to j}$ is the coordinate of $\mathbf{c}$ indexed by the edge $i \to j$.

Note that this definition is consistent with the original one in the case that the nonbacktracking matrix of G is irreducible. Indeed, in this case the Perron vector $\mathbf{v}$ of $B(0)^t$ can be unambiguously chosen. We just need to prove that it coincides with our $\mathbf{c}$. But since the $\rho(B(0)^t)$-eigenspace is one-dimensional and both $\mathbf{c}$ and $\mathbf{v}$ are of norm one and non-negative we must necessarily have $\mathbf{c} = \mathbf{v}$.

Example 4.4. *With this definition the nonbacktracking centrality of the graph G in Figure 4 can be determined. In this example the matrices $B(\alpha)$ and $B(0)$ are as follows, where we are using the lexicographical order for the edges, that is,* $(1 \to 2,\ 2 \to 1,\ 2 \to 3,\ 3 \to 2)$*:*

$$B(\alpha) = \begin{pmatrix} 0 & \alpha & 1 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 1 & \alpha & 0 \end{pmatrix} \qquad B(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the graphs corresponding to $B(\alpha)$ and $B(0)$ are those in figures 5 and 6 respectively.

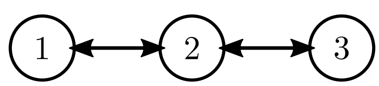

Figure 4: An example graph G with 3 nodes

We get, through direct computation, that the spectral radius $\rho_\alpha = \rho(B(\alpha)^t) = \sqrt{\alpha^2+\alpha}$ and that $\mathbf{v}(\alpha) = (\alpha, \sqrt{\alpha^2+\alpha}, \sqrt{\alpha^2+\alpha}, \alpha)$ is a positive eigenvector of $B(\alpha)^t$ associated to ρ_α. Therefore

$$\mathbf{c}(\alpha) := \mathbf{v}(\alpha)/||\mathbf{v}(\alpha)||_1$$

is the Perron vector of $B(\alpha)^t$. It is easy to prove that

$$\mathbf{c} = \lim_{\alpha\downarrow 0} \mathbf{c}(\alpha) = (0, 1/2, 1/2, 0)$$

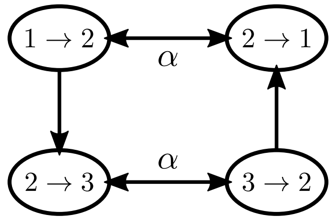

Figure 5: The graph associated to $B(\alpha)$

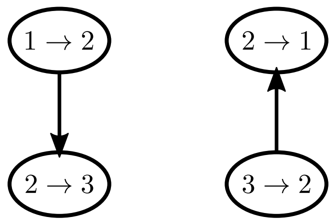

Figure 6: The graph associated to $B(0)$

and this vector is what we have defined as the nonbacktracking edge centrality of G*. Note that in this case* $\rho(B(0)^t) = 0$ *and the associated eigenspace* $V_0 = \{(0, \mu, \nu, 0)\}_{\mu,\nu\in\mathbb{R}}$ *has dimension 2.*

Finally, for the nonbacktracking centrality of the nodes of G*, we have*

$$c_1 = c_{1\to2} = 0, \qquad c_2 = c_{2\to1} + c_{2\to3} = 1, \qquad c_3 = c_{3\to2} = 0.$$

Remark 4.5. Next we point out how Theorem 4.1 can be used to study the limit case of other spectral centrality measures. For instance, when $M_0 = A^t$, where A is the adjacency matrix of directed graph G (which is not necessarily strongly connected), and M is the matrix with every entry $m_{ij} = 1$, the Perron eigenvector $\mathbf{c}(\alpha)$ of M_α is the α-centrality of G ,[6] and Theorem 4.1 proves that this family of vectors converge when $\alpha \downarrow 0$.

As another example, we can consider the PageRank of G [7], indexed by the damping factor q, which is defined as the Perron eigenvector of the matrix

$$R_q = qP^t + (1-q)N^t,$$

where P is obtained by normalization of each row of the adjacency matrix A and N is the personalization matrix (which can be seen as a generalization of the personalization vector). If we set $\alpha = 1-q$, $M_0 = P^t$ and $M = N^t - P^t$ we get $M_\alpha = M_0 + \alpha M = R_q$. Therefore Theorem 4.1 proves the convergence of the PageRank vector when $q \uparrow 1$. This result has been shown by Boldi et al. in Section 5 of [4] for the case of a matrix N constructed from a personalization vector. Our theorem provides an alternative proof and generalizes the result to the case of an arbitrary personalization matrix. Note that the limit cases of eigenvector-type centralities have been studied in the literature (see, for example [1, 25]) and Theorem 4.1 proves the existence of such limit cases.

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