Asymptotic Discord and Entanglement of Non-Resonant Harmonic Oscillators in an Equilibrium Environment

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In this work, we calculate the exact asymptotic quantum correlations between two interacting non-resonant harmonic oscillators in a common Ohmic bath. We derive *analytical formulas* for the covariances, fully describing any Gaussian stationary state of the system, and use them to study discord and entanglement in the strong and weak dissipation regimes. We discuss the rich structure of the discord of the stationary separable states arising in the strong dissipation regime. Also under strong dissipation, when the modes are not mechanically coupled, these may entangle only through their interaction with the *common* environment. Interestingly enough, this stationary entanglement is only present within a *finite band of frequencies* and increases with the dissipation rate. In addition, robust entanglement between *detuned* oscillators is observed at low temperature.

Over the last decades, non-classical correlations such as *discord* and *entanglement* have been widely acknowledged to play a central role in quantum mechanics. In particular, the robustness of entanglement against decoherence in continuous-variable (CV) systems has been object of study in a number of recent publications [1–10]. Due to the fact that entanglement is a valuable resource in quantum communication and information processing with CV, the investigation of the conditions under which bipartite states *indefinitely* preserve their non-separable nature in presence of noise and dissipation is not only of fundamental importance, but also of practical interest.

Gaussian two-mode states occupy a priviledged position among all entangled CV preparations, since they combine easy experimental realization (e.g. the output beams in an optical parametric oscillator), with simple mathematical description in terms of their second order moments, which are experimentally determined via homodyne detection [11]. These CV systems also find physical realization in mechanical oscillators, such as trapped ions coupled by their electrostatic interaction [12]. A simple but rather general model describing those systems consists of two coupled detuned harmonic oscillators, linearly interacting, and in contact with a bath.

The stationary entanglement of two resonant modes in a squeezed state under decoherence was studied in Refs. [2], whereas dynamical features of the non-resonant case were addressed in [3]. Additionally, different *long-time* behaviours were experimentally observed in [13] after simulating decoherence on one of the modes. Other recent publications such as [7, 8] treat the dynamics of quantum correlations in similar systems for a variety of structured environments. Nonetheless, it must be noted that the majority of these works rely on *weak* system-environment coupling assumption.

On reference to quantum discord [14], it was introduced to capture the *quantumness* of correlations. Since the majority of quantum states, including most of the separable ones, have non-zero discord [15], it is said to be more general than entanglement as a measure of non-classicality. Discord has recently attracted attention in the field of quantum information, now that for some computational models wich make no use of entanglement, it is believed to provide exponential speedup over their best known classical counterparts [16].

In this Letter, we provide *exact analytical formulas* fully characterizing the Gaussian stationary state of two interacting non-resonant modes in contact with a common thermal environment by solving the quantum Langevin equations in the asymptotic limit [4, 5]. We then compute quantum discord and entanglement exactly, being our main interest the robustness of stationary quantum correlations against temperature and dissipation, in the less studied case of finitely *detuned* modes.

Our system consists of two harmonic oscillators coupled through a term bilinear in the coordinates $H_S = \sum_{i=1}^{2} \frac{p_i^2}{2} + \frac{1}{2}\omega_i^2 x_i^2 + \frac{k}{2}(x_1 - x_2)^2$. The interaction with a thermal environment can be modeled by coupling them to a common bosonic heat bath, which yields: $H = H_S + \sum_\mu \left(\frac{p_\mu^2}{2m_\mu} + \frac{1}{2}m_\mu \omega_\mu^2 x_\mu^2 \right) + \mathcal{F}(x_1, x_2) \sum_\mu g_\mu x_\mu + \mathcal{F}(x_1, x_2)^2 \Omega^2/2$ [17], where Ω^2 is given by $2\int_0^\infty d\omega \, J(\omega)/\omega$, and $J(\omega)$ is the bath spectral density [18]. We use units in which $\hbar = k_B = 1$.

As noted in Refs. [2, 7] among others, under suitable conditions, an initially separable state may become entangled in this model, even if $k = 0$. However, this ability of the *common* environment to create non-classical correlations between the two parties highly depends on the physical distance r that separates them [4, 10], so that *distant* dissipative oscillators would be better modeled by considering *independent* identical environments.

In the following, all averages are taken with respect to a separable initial state in the system's and bath's degrees

of freedom $\rho(0) \otimes \rho_T$ where $\rho(0)$ is a Gaussian state of the two oscillators, and ρ_T is a thermal equilibrium state of the environment at temperature T.

By choosing $\mathcal{F}(x_1, x_2) = x_1 + x_2$ in H one may derive the following quantum Langevin equation [18]:

$$\ddot{x}_i + \left(\omega_i^2 + \Omega^2\right) x_i + \sum_{j=1}^{2}\left[k\left(x_i - x_j\right) + \Omega^2 x_j\right]\tilde{\delta}_{ij} \quad (1)$$

$$= F(t) + \sum_{i=1}^{2}\int_{-\infty}^{t} ds\ \chi(t-s)\ x_i(s),$$

where $\tilde{\delta}_{ij} \equiv 1 - \delta_{ij}$. The quantum force $F(t)$ acting on both oscillators and the dissipative kernel $\chi(t)$ are connected by the Kubo relation $\chi(t) = i\theta(t)\langle[F(t), F(0)]\rangle$, where $\theta(t)$ stands for the Heaviside step function. For this choice of $\mathcal{F}(x_1, x_2)$, the overall system Hamiltonian H is quadratic in positions and momenta, so that for $\omega_1 \neq \omega_2$ the asymptotic state (which is independent of $\rho(0)$) is guaranteed to be Gaussian. In the case of resonant oscillators, the normal mode decomposition leads to an effective decoupling of the relative degree of freedom from the bath, and thus the stationary state would depend on the initial condition [2]. In the following we shall restrict ourselves to non-resonant oscillators.

We can solve Eq. (1) by Fourier transforming and denoting $\hat{f}(\omega) \equiv \int_{-\infty}^{\infty} dt\ e^{i\omega t}f(t)$. Due to the linearity of the problem, this formally yields: $\hat{x}_i(\omega) = \sum_j \alpha_{ij}(\omega)\hat{F}(\omega)$, where the generalized susceptibilities $\alpha_{ij}(\omega)$ are elements of the inverse of the matrix: $(\alpha^{-1})_{ii} = \omega_i^2 + \Omega^2 - \omega^2 + k - \hat{\chi}(\omega)$ and $(\alpha^{-1})_{ij} = -k + \Omega^2 - \hat{\chi}(\omega)$ for $i \neq j$.

The central object of our study is the asymptotic covariance matrix $\mathbf{\Gamma}$, defined as $\Gamma_{ij} \equiv \langle R_i R_j \rangle - [R_i, R_j]/2$, with $\mathbf{R} \equiv \{x_1, p_1, x_2, p_2\}$ which, up to local displacements, fully characterizes any Gaussian state of our bipartite system. Provided with the susceptibility matrix $\alpha(\omega)$, we may compute the power spectrum $\langle x_i x_j \rangle_\omega \equiv \int_{-\infty}^{\infty} dt\ e^{i\omega(t-t')}\langle x_i(t-t')x_j(0)\rangle$ of the stationary two-time correlation $\langle x_i(t-t')x_j(0)\rangle = \langle x_i(t)x_j(t')\rangle$ in terms of the power spectrum of the environmental force:

$$\langle x_i x_j \rangle_\omega = \sum_{k,l} \alpha_{ik}(\omega)\alpha_{jl}(-\omega)\langle FF \rangle_\omega. \quad (2)$$

The rest of power spectra $\langle R_i R_j \rangle_\omega$, can be readily obtained from Eq. (2) since $\langle p_i x_j \rangle_\omega = -i\omega\langle x_i x_j \rangle_\omega$ and $\langle p_i p_j \rangle_\omega = \omega^2\langle x_i x_j \rangle_\omega$. The full asymptotic covariance matrix then reads $\Gamma_{ij}(t,t) = \int_{-\infty}^{\infty}(d\omega/2\pi)\ \langle R_i R_j \rangle_\omega - [R_i, R_j]/2$.

To proceed further in deriving analytical expressions, we shall choose an Ohmic spectral density with Lorentz-Drude cutoff $J(\omega) = 2\gamma\omega/\pi(1 + \omega^2/\omega_c^2)$, that leads to a dissipative kernel $\hat{\chi}(\omega) = 2\gamma\omega_c^2/(\omega_c - i\omega)$, and a bath correlation function $\langle FF \rangle_\omega = \pi J(\omega)[1 + \coth(\omega/2T)]$, where ω_c is the cutoff frequency and γ the dissipation rate. The renormalization constant appearing in Eq. (1)

is $\Omega^2 = 2\gamma\omega_c$ for this choice of $J(\omega)$. We start from Eq. (2) to get

$$\langle x_i(t)x_j(t)\rangle = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{g_{ij}(\omega)\coth\left(\frac{\omega}{2T}\right)}{h(\omega)h(-\omega)}, \quad (3)$$

where $g_{ij}(\omega)$ and $h(\omega)$ are 5th order polynomials depending on $\omega_1, \omega_2, \omega_c, \gamma$ and k. After a lengthly calculation, these integrals can be solved to yield [19]:

$$\langle x_i(t)x_j(t)\rangle = \langle x_i x_j \rangle_{T\to\infty} - \sum_{k=1}^{5} a_k^{(i,j)}\psi\left(1 - \frac{iz_k}{2\pi T}\right)$$

$$\langle p_i(t)p_j(t)\rangle = \langle p_i p_j \rangle_{T\to\infty} - \sum_{k=1}^{5} b_k^{(i,j)}\psi\left(1 - i\frac{z_k}{2\pi T}\right),$$

while the remaining convariances are all zero. $a_k^{(i,j)}$ and $b_k^{(i,j)}$ are temperature-independent coefficients for which we have explicit formulas, z_k are the five complex roots of $h(\omega)$, and ψ stands for the logarithmic derivative of Euler's gamma function. $\langle R_i R_j \rangle_{T\to\infty}$ denotes *classical* mean values, to which the summations are *quantum corrections*.

To gain some insight into the physics of the problem, we will set $k = 0$ and place ourselves in the limit of $\gamma/\omega_c \ll 1$. To first order in this small parameter, the roots z_k become $\pm\omega_i \pm \frac{\gamma\omega_c}{(\omega_i \mp i\omega_c)}$ and $i\omega_c - i2\gamma\omega_c^2\left(\omega_1^2 + \omega_2^2 + 2\omega_c^2\right)/\prod_i(\omega_i^2 + \omega_c^2)$. Additionally, in the very low temperature regime (i.e. $T \ll \omega_i$) the asymptotic expansion $\psi(1 + z) \simeq \log z + (2z)^{-1}$ may be justified. Inserting all this into the formulas of the Γ_{ij}, and retaining terms only to first order in T/ω_c, ω_i/ω_c and γ/ω_c one gets the temperature-independent expressions:

$$\langle x_i^2 \rangle \simeq \frac{1}{2\omega_i} - \frac{\gamma}{2\pi}\left[\frac{2\omega_c - \pi\omega_i}{\omega_i^2\omega_c} + \frac{4}{\omega_c^2}\left(\log\frac{\omega_c}{\omega_i}\right) - \frac{1}{2}\right]$$

$$\langle p_i^2 \rangle \simeq \frac{1}{2}\omega_i + \frac{\gamma}{2\pi}\left[3\pi\frac{\omega_i}{\omega_c} - 2 + 4\log\frac{\omega_c}{\omega_i}\right]$$

$$\langle x_1 x_2 \rangle \simeq \frac{\gamma}{\pi\left(\omega_1^2 - \omega_2^2\right)}2\log\frac{\omega_2}{\omega_1}$$

$$\langle p_1 p_2 \rangle \simeq \frac{\gamma\log 16}{2\pi} + \frac{\gamma\left(\omega_1^2\log\frac{\omega_c^2}{4\omega_1^2} - \omega_2^2\log\frac{\omega_c^2}{4\omega_2^2}\right)}{\pi\left(\omega_1^2 - \omega_2^2\right)}.$$

By setting $\gamma = 0$, the variances $\langle x_i^2 \rangle$ and $\langle p_i^2 \rangle$ become those of the ground state of the corresponding oscillators without dissipation, and the asymptotic (separable) state has a diagonal covariance matrix. Taking into account that $\omega_c/\omega_i \gg 1$, one sees that increasing γ leads to a reduction of $\langle x_i^2 \rangle$ while $\langle p_i^2 \rangle$ gets larger, as expected. The terms proportional to $\log\omega_c/\omega_i$ in $\langle p_i p_j \rangle$, also occur in the same region of parameters for a single damped oscillator [18].

These analytical and *exact* expressions for the asymptotic covariances Γ_{ij} are one of the main results of this Letter. From them, we shall compute all quantum correlations without *any assumptions* on the different time
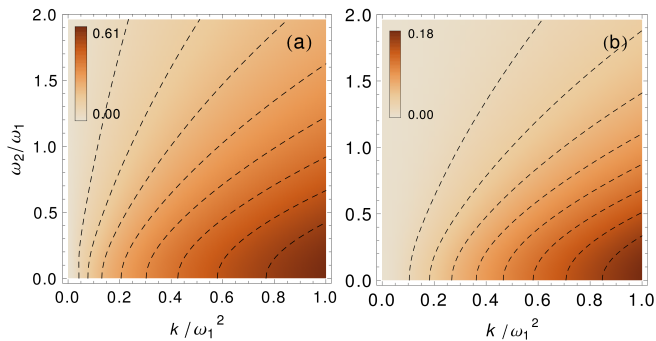
FIG. 1. (a) Asymptotic entanglement and (b) discord in the weak coupling regime as functions of k and ω_2. We have chosen $\omega_1 = 10$, $\gamma = 0.01$, $\omega_c = 500$ and $T = 0.5$. Both measures of quantum correlations exhibit a similar behavior: They increase with $\omega_2/k \to 0$. Dashed lines are contours and the color legends are plotted in the upper left corners.

FIG. 2. (a) Asymptotic entanglement in the strong dissipation regime as a function of k and ω_2. Due to the bigger decoherence entanglement is mostly lost, though its maximum is still attained as $\omega_2/k \to 0$. Interestingly, in the inset of (a), we can observe a small region around $k = 0$ where non-classical correlations are *enhanced* by the dissipation. This purely non-Markovian effect, enters into our equations through the renormalization term $\Omega^2 x_1 x_2$ in the Hamiltonian H, which for sufficiently high dissipation rate, results in an effective environment-assisted coupling between the oscillators. (b) Asymptotic entanglement (continuous line) and discord (dashed line) as functions of ω_2 alone for $k = 0$. For both plots, parameters are the same as in Fig. 1 except for the dissipation $\gamma = 0.5$.

scales involved in the problem. The *logarithmic negativity* [20] is a suitable measure of entanglement is this case. It is based on the *positivity of the partial transpose* separability criterion [21], which is necessary and sufficient for bipartite Gaussian states [22]. When it comes to discord, it is essentially the difference between the mutual information of the bipartite state, before and after performing a complete measurement on one of the parts (in what follows, mode two). Even though its calculation is usually very involved, if one restricts to Gaussian generalized measurements it is straightforward to compute for bipartite Gaussian states, as recently shown in [8, 23, 24]. We now turn to present the results thus obtained:

Weak dissipation.— We will look first into the weak dissipation regime, where the relaxation time scale $\tau_R = \gamma^{-1}$ is much larger than the bath correlation time, given by $\tau_B = \nu_1^{-1}$ whenever $\omega_c > \nu_1$ [25]. We shall also pick a low temperature T for which non-zero asymptotic entanglement might exist. Results are presented in Fig. 1. In accordance with [2], we observe $E_\mathcal{N} \neq 0$ *out of resonance* at low temperatures, and overall similar behaviour of entanglement and discord.

In the limit $k \gg \omega_i$, the oscillators, whose effective frequency $\tilde{\omega}^2$ is given by $\omega_i^2 + k + 2\gamma\omega_c$, become nearly resonant and thus, approximately decouple in the variables $\eta_\pm \equiv (x_1 \pm x_2)/\sqrt{2}$ and $\pi_\pm \equiv (p_1 \pm p_2)/\sqrt{2}$. As $\omega_i/k \to 0$, $\{\langle\eta_+^2\rangle, \langle\pi_-^2\rangle\} \to \infty$ while $\{\langle\pi_+^2\rangle, \langle\eta_-^2\rangle\} \to 0$ [5]. This corresponds to a non-symmetric two-mode squeezed thermal state with infinite squeezing r, i.e. the ideal EPR (maximally entangled) state [26].

Strong dissipation.— We achieve this regime by increasing γ, so that the relaxation time τ_R becomes comparable to the bath correlation time τ_B. Results are presented and summarized in Fig. 2.

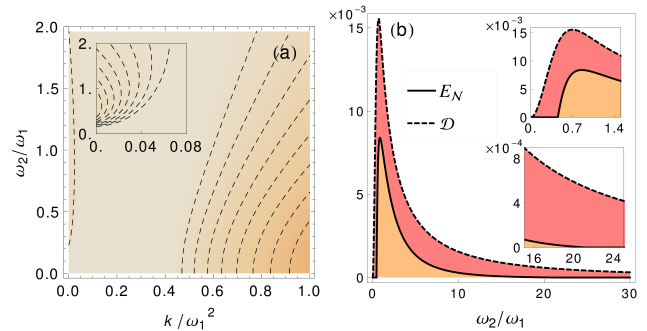Even at $k = 0$, there may exist non-vanishing stationary entanglement. The corresponding asymptotic states are thus non-Gibbsian, as could be expected at sufficiently low temperatures [27]. Actually, one sees in Fig. 2 (b) that for fixed ω_1, γ, T and ω_c, there exists a *band* of frequencies $\omega_2 \in [\omega', \omega'']$ for which the asymptotic state is non-separable. As the temperature increases, the amount of entanglement gets reduced, while ω' and ω'' come closer to each other. On the contrary, by increasing the dissipation rate γ, the maximum attainable $E_\mathcal{N}$ and the bandwidth $\omega'' - \omega'$ get larger. Reducing the frequency ω_1 also leads to narrower bandwidth. One may also see that ω' is almost insensitive to changes in ω_c, while ω'' slightly decreases as ω_c grows.

The simple model $H_S = \sum_i[p_i^2/2 + (\omega_i^2 + \Omega^2)x_i^2/2] + \Omega^2 x_1 x_2$ proves useful to understand these features. Note that the renormalization $\Omega^2 = 2\gamma\omega_c$ has been introduced to account for the bath-mediated coupling. If $w_i/\Omega^2 \ll 1$, the oscillators are again near resonance and approximately decouple in the coordinates $\{\eta_\pm, \pi_\pm\}$. Then, $E_\mathcal{N} \neq 0$ whenever $\tilde{\nu}_-^2 = \langle\eta_-^2\rangle\langle\pi_+^2\rangle < 1/4$ [5].

It is easy to see that the lowest symplectic eigenvalue of the partially transposed covariance matrix $\tilde{\nu}_-$ grows with T and decreases with Ω^2 and ω_i. Thus, an increase in the temperature must be compensated by an increase in ω' for the inequality to be satisfied. Conversely, an increase in either Ω^2 or ω_1 allows for a reduction of the critical frequency ω'. The existence of an upper bound ω'' for this non-separable region is related to the structure of the bath, so that its behavior can not be inferred from this simple non-dissipative model.

Finally, we also remark that in our symplified model, $\tilde{\nu}_-$ decreases as $|\Omega^2 - k|$ grows, since the effective coupling
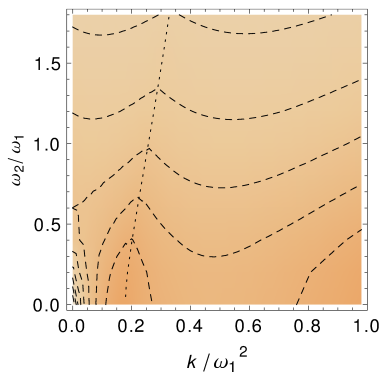
FIG. 3. The asymptotic discord displays a very different behaviour in the strong dissipation regime. Since the entanglement has considerably decreased, it no longer dominates over other quantum correlations and, as could be expected, remains non-zero even when the asymptotic state is separable. It displays a maximum around $k \simeq 20$, marked with the dotted line, and also vanishes in the limit $\{k, \omega_2\} \to 0$ (see text for discussion). All parameters are the same as in Fig. 2.

between the oscillators is the net result of two competing effects, that may eventually cancel each other: The environment-assisted coupling and the direct interaction between the oscillators. This explains qualitatively why $E_{\mathcal{N}} = 0$ for *intermediate* couplings in Fig. 2.

Finally, Fig. 3 contains results on discord in this regime. The global purity μ may be found to be roughly constant for all stationary states, except those with $\{k, \omega_2\} \to 0$, while $\tilde{\nu}_-$, is maximum on the dotted line of figure 3. This suggest that at fixed μ, Gaussian quantum discord increases with $\tilde{\nu}_-$, which is consistent with [23], where this was seen to hold for squeezed thermal states. Additionally, since the partial purity $\mu_2 \to 0$ as $\{k, \omega_2\} \to 0$ the marginal stationary states $\rho_2 \equiv \mathrm{Tr}_1(\rho_{12})$ become maximally mixed in this limit, therefore yielding a quantum-classical state [28] with zero discord as revealed by measurements on mode two.

At higher temperatures and stronger dissipation rates, the fate of the asymptotic entanglement is to disappear completely, in contrast with the asymptotic quantum discord, which is always non-zero as could be expected [15].

To summarize and conclude, we have exactly solved the problem of the asymptotic quantum correlations in a system of two coupled non-resonant harmonic oscillators in a common bath. We provided exact analytical formulas for the covariances, fully characterizing any Gaussian stationary state of the system, and used them to compute the asymptotic entanglement and discord, in the weak and strong dissipation regimes. We found that stationary entanglement between non-resonant oscillators may exist at sufficiently low temperatures. Furthermore, for a finite band of frequencies, it may be created only due to the environment-assited interaction between the two modes in the strong dissipation regime. Also in this regime we

discuss the non trivial structure that quantum discord shows when the stationary states are separable.

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